RESULTS ON FOURIER MULTIPLIERS

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Abstract. The problem of giving necessary and sufficient conditions for Fourier multipliers to be bounded on L^p spaces does not have a satisfactory answer for general p. In this paper, some of the facts that are known about Fourier multipliers are detailed. The starting point is a quick tour of singular integral theory, leading into the Mikhlin multiplier theorem. An important application to Littlewood-Paley theory and a proof C. Fefferman's ball multiplier theorem is presented.

Contents

1. Introduction

A Fourier multiplier is a function m from $\mathbb{R}^d$ to $\mathbb{C}$ that defines an operator T_m through multiplication on a function's frequency spectrum. In particular, the operator T_m is defined on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ by $T_m f = (m\hat{f})^\vee$, where

$$\hat{f}(\xi) = (f)^\wedge(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x}\,\mathrm{d}x$$

and

$$\check{f}(x) = (f)^\vee(x) = \int_{\mathbb{R}^d} f(\xi) e^{2\pi i x \cdot \xi}\,\mathrm{d}\xi$$

are the Fourier transform and inverse Fourier transform of the function f, respectively. It is natural to seek conditions on m under which T_m can be extended to a bounded operator on L^p. For general p, there is no known characterization of such m, but the Mikhlin multiplier theorem provides convenient sufficient conditions. The majority of this paper is dedicated to proving this theorem and an important application of it. The ball multiplier theorem, which says that T_m is unbounded for $p \neq 2$ and $d > 1$ when m is the characteristic function of the unit ball, is presented at the end to show the difficulty of a general theory. Theorem 2.8 will use the

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fundamental Calderón-Zygmund decomposition of a function, which can be found in [1, p. 44]. Most of the proofs in this paper are influenced by those in [1] and [2].

2. Calderón-Zygmund Operators

The goal of this section is to prove the boundedness of a class of operators on L^p for $1 < p < \infty$. This result will be used to prove an important theorem on Fourier multipliers.

Definition 2.1. A Calderón-Zygmund kernel is a function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ that satisfies

(K1) $|K(x)| \le B|x|^{-d}$ for all $x \in \mathbb{R}^d$
(K2) $\int_{[|x|>2|y|]} |K(x) - K(x-y)| \,\mathrm{d}x \le B$ for all $y \neq 0$
(K3) $\int_{[r<|x|<s]} K(x) \,\mathrm{d}x = 0$ for all $r, s > 0$

for some constant B. The Calderón-Zygmund operator T associated with the kernel K is defined by

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{[|x-y|>\varepsilon]} K(x-y) f(y) \,\mathrm{d}y \tag{2.2}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Note that the limit in (2.2) exists for all x. Indeed, using (K3) followed by (K1) we have

$$\begin{aligned}\left| \int_{[1>|x-y|>\epsilon]} K(x-y) f(y) \,\mathrm{d}y \right| &= \left| \int_{[1>|y|>\epsilon]} K(y)(f(x-y) - f(x)) \,\mathrm{d}y \right| \\ &\le B \left(\sup |\nabla f(y)| \right) \int_{[1>|y|>\epsilon]} \frac{\mathrm{d}y}{|y|^{d-1}},\end{aligned}$$

and the part of the integral with $|x - y| > 1$ is easily controlled using (K1) and Schwartz bounds on f. Note how the limit in (2.2) and the smoothness of f allows us to use the cancellation property in (K3). The following result allows one to check that (K2) holds for a kernel given a bound on the derivative.

Theorem 2.3. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfy $|\nabla K(x)| \le B|x|^{-d-1}$ for all $x \neq 0$ and some $B > 0$. Then*

$$\int_{[|x|>2|y|]} |K(x) - K(x-y)| \,\mathrm{d}x \le CB, \tag{2.4}$$

for all $y \neq 0$ and $C = C(d)$ a constant depending only on d, and therefore K satisfies (K2).

Proof. Fix $x, y \in \mathbb{R}^d$ with $|x| > 2|y|$. Define $\gamma : [0,1] \to \mathbb{R}^d$ by $\gamma(t) = (1-t)(x-y) + tx$. Then the image of γ is the line segment connecting $x - y$ and x and is contained in $B(x, |x|/2)$. We have

$$\begin{aligned}|K(x) - K(x-y)| &= \left| \int \nabla K \cdot \mathrm{d}\gamma \right| = \left| \int_0^1 (\nabla K)(x - (1-t)y) \cdot y \,\mathrm{d}t \right| \\ &\le \int_0^1 |((\nabla K)(x - (1-t)y)| |y| \,\mathrm{d}t \le B 2^{d+1} |x|^{-d-1} |y|.\end{aligned}$$

Using this estimate and

$$\int_{[|x|>2|y|]} \frac{1}{|x|^{d+1}}\,\mathrm{d}x \leq C(d)\frac{1}{|x|} \leq C(d)\frac{1}{|y|}$$

on (2.4) finishes the proof. □

The Hilbert transform is given by a Calderón-Zygmund operator with kernel $1/x$. More generally, any kernel of the form $\Omega(x/|x|)|x|^{-d}$, where Ω is a smooth function on the unit sphere with mean value 0, defines a Calderón-Zygmund operator.

Conditions (K1)-(K3) are sufficient to prove that T can be extended to a bounded operator on L^2. Together with a weak-L^1 bound, Marcinkiewicz interpolation, and a duality argument, this shows that T is in fact bounded on L^p for $1 < p < \infty$, failing at the endpoints. Since the weak-L^1 bounds do not require (K3), it is often desirable to consider kernels for which (K3) does not necessarily hold, instead taking as hypothesis boundedness on L^2. Then L^p boundedness holds for this larger class of operators.

The failure of boundedness at $p = 1$ can be seen qualitatively by considering the action of T on the Dirac delta distribution δ_0. In particular, we have $T\delta_0 \simeq 1/|x|^d \notin L^1(\mathbb{R}^d)$. Formally, one approximates δ_0 by an approximate identity and uses a limiting process.

Theorem 2.5. *The operator T associated with a kernel K as in Definition 2.1 can be extended to a bounded operator on L^2. Specifically $\|Tf\|_{L^2} \leq CB\|f\|_{L^2}$ for all $f \in \mathcal{S}$, where $C = C(d)$ depends only on the dimension d.*

Proof. Let K be as in Definition 2.1. We write $\chi_{[r<|y|<s]}$ to represent the characteristic function of the annulus with inner and outer radii r and s. Define

$$(T_{r,s}f)(x) = \int_{\mathbb{R}^d} K(y)\chi_{[r<|y|<s]}(y)f(x-y)\,\mathrm{d}y$$

for $0 < r < s < \infty$. Let $m_{r,s}$ be the Fourier transform of the restricted kernel $y \mapsto K(y)\chi_{[r<|y|<s]}(y)$. By Plancherel's theorem, we know

$$\left\|T_{r,s}f\right\|_{L^2} = \left\|(K\chi_{[r<|y|<s]} * f)^\wedge\right\|_{L^2} = \left\|m_{r,s}\hat{f}\right\|_{L^2} \leq \left\|m_{r,s}\right\|_{L^\infty}\|f\|_{L^2},$$

and so it suffices to prove $\sup_{0<r<s}\left\|m_{r,s}\right\|_{L^\infty} \leq CB$; if this holds, then the pointwise identity $Tf(x) = \lim_{r\to 0, s\to\infty}(T_{r,s}f)(x)$ and Fatou's lemma finish the proof. To prove this L^∞ bound on the Fourier transform, we split the integral as follows

$$\left|m_{r,s}(\xi)\right| \leq \left|\int_{[r<|x|<|\xi|^{-1}]} e^{-2\pi i x\cdot\xi}K(x)\,\mathrm{d}x\right| + \left|\int_{[|\xi|^{-1}<|x|<s]} e^{-2\pi i x\cdot\xi}K(x)\,\mathrm{d}x\right|. \tag{2.6}$$

Using (K3) followed by (K1) on the first integral, we have

$$\begin{aligned}\left|\int_{[r<|x|<|\xi|^{-1}]} e^{-2\pi i x\cdot\xi}K(x)\,\mathrm{d}x\right| &= \left|\int_{[r<|x|<|\xi|^{-1}]} (e^{-2\pi i x\cdot\xi}-1)K(x)\,\mathrm{d}x\right| \\ &\leq \left|\int_{[|x|<|\xi|^{-1}]} 2\pi|x||\xi|\left|K(x)\right|\,\mathrm{d}x\right| \\ &\leq 2\pi|\xi|\int_{[|x|<|\xi|^{-1}]} B|x|^{-d+1}\,\mathrm{d}x \leq CB.\end{aligned}$$

The second integral bound uses condition (K2) and cancellation properties of $e^{-2\pi i x\cdot\xi}$. We have

$$F := \int_{[|\xi|^{-1}<|x|<s]} K(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x = -\int_{[|\xi|^{-1}<|x|<s]} K(x)e^{-2\pi i\left(x+\xi/(2|\xi|^2)\right)\cdot\xi}\,\mathrm{d}x$$
$$= \int_{[|\xi|^{-1}<\left|x-\xi/(2|\xi|^2)\right|<s]} K\left(x-\frac{\xi}{2|\xi|^2}\right)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x$$

and therefore

$$2F = \int_{[|\xi|^{-1}<|x|<s]} \left(K(x) - K\left(x-\frac{\xi}{2|\xi|^2}\right)\right)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x + R \tag{2.7}$$

where R is defined appropriately. The first term in (2.7) is controlled in absolute value via (K2). The R represents the difference in the integration regions $[|\xi|^{-1} < |x| < s]$ and $[|\xi|^{-1} < \left|x-\xi/(2|\xi|^2)\right| < s]$. We can use (K1) to prove $|R| < CB$. Specifically, the difference in integration regions can be split into four parts:

$$|\xi|^{-1} < |x| < s \text{ and } \left|x-\xi/(2|\xi|^2)\right| \le |\xi|^{-1}$$
$$|\xi|^{-1} < |x| < s \text{ and } s \le \left|x-\xi/(2|\xi|^2)\right|$$
$$|\xi|^{-1} < \left|x-\xi/(2|\xi|^2)\right| < s \text{ and } |x| \le |\xi|^{-1}$$
$$|\xi|^{-1} < \left|x-\xi/(2|\xi|^2)\right| < s \text{ and } s \le |x|\,.$$

Noting that the first region is a subset of the region $[|\xi|^{-1}/2 \le \left|x-\xi/(2|\xi|^2)\right| \le |\xi|^{-1}]$, condition (K1) easily shows the integral over the region to be less than CB. A similar argument works for the other three regions. This finishes the proof. $\square$

We now prove that T has a weak-L^1 bound via the Calderón-Zygmund decomposition.

Theorem 2.8. *Let T be a linear operator that is bounded on $L^2(\mathbb{R}^d)$ such that*

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x-y)f(y)\,\mathrm{d}y$$

for all $f \in L^2$ with compact support and for all $x \notin supp(f)$, and such that K satisfies (K2) with constant B. Then for all $f \in \mathcal{S}(\mathbb{R}^d)$ we have the weak-L^1 bound

$$\left|\left\{x\in\mathbb{R}^d : \left|Tf(x)\right| > \lambda\right\}\right| \le \frac{CB}{\lambda}\|f\|_{L^1}$$

for all $\lambda > 0$, where C depends only on the dimension d.

Proof. By dividing K by B, we can assume without loss of generality that $B = 1$. Fix $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$. Using the Calderón-Zygmund decomposition of a function with height λ, we can write $f = g + b$, where g is the 'good' and b is the 'bad' part of f. The decomposition gives g and b with the following properties (see [1, p. 44] for proof). There is a collection $\mathcal{B}$ of disjoint cubes for which $b = \sum_{Q\in\mathcal{B}} \chi_Q f$. We have $\left|\cup_{Q\in\mathcal{B}} Q\right| < \lambda^{-1}\|f\|_{L^1}$, and for any $Q \in \mathcal{B}$ we have

$$\lambda < \frac{1}{|Q|}\int_Q |f| \leq 2^d \lambda.$$

Here $|\cdot|$ is Lebesgue measure. Finally, we have $|g| \leq \lambda$.

Define

$$f_1 = g + \sum_{Q\in\mathcal{B}} \chi_Q \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$$
$$f_2 = b - \sum_{Q\in\mathcal{B}} \chi_Q \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x = \sum_{Q\in\mathcal{B}} f_Q$$

where

$$f_Q = \chi_Q \left(f - \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x \right).$$

Clearly $f = f_1 + f_2$. From $|g| < \lambda$, $\int_Q |f| \leq |Q|\, 2^d \lambda$, and the cubes and $\mathrm{supp}(g)$ are disjoint, we conclude $\|f_1\|_{L^\infty} \leq 2^d\lambda$. From the definition of f_Q it is clear that $\|f_2\|_{L^1} \leq 2\|f\|_{L^1}$, and a similar argument shows that $\|f_1\|_{L^1} \leq \|f\|_{L^1}$. Finally it is clear that the f_Q have mean value zero; that is

$$\int_Q f_Q(x)\,\mathrm{d}x = 0.$$

Using Chebyshev's inequality, we have

$$\begin{aligned}
\left|\left\{x\in\mathbb{R}^d : \left|(Tf)(x)\right| > \lambda\right\}\right| &\leq \left|\left\{x\in\mathbb{R}^d : \left|(Tf_1)(x)\right| > \frac{\lambda}{2}\right\}\right| \\
&\quad + \left|\left\{x\in\mathbb{R}^d : \left|(Tf_2)(x)\right| > \frac{\lambda}{2}\right\}\right| \\
&\leq \frac{C}{\lambda^2}\|Tf_1\|_{L^2}^2 + \left|\left\{x\in\mathbb{R}^d : \left|(Tf_2)(x)\right| > \frac{\lambda}{2}\right\}\right|.
\end{aligned} \tag{2.9}$$

Since we assumed T bounded on L^2, the first term on the last line of (2.9) is bounded:

$$\frac{C}{\lambda^2}\|Tf_1\|_{L^2}^2 \leq \frac{CB}{\lambda^2}\|f_1\|_{L^2}^2 \leq \frac{CB}{\lambda^2}\|f_1\|_{L^\infty}\|f_1\|_{L^1} \leq \frac{CB}{\lambda}\|f_1\|_{L^1}.$$

So it only remains to estimate the second term. Define Q^* to be the dilate of Q by $2\sqrt{d}$ for any $Q\in\mathcal{B}$ (that is, Q^* and Q share a center and Q^* has edges longer than Q by a factor of $2\sqrt{d}$). Then

$$\begin{aligned}
\left|\left\{x\in\mathbb{R}^d : \left|(Tf_2)(x)\right| > \frac{\lambda}{2}\right\}\right| &\leq \left|\cup_{\mathcal{B}} Q^*\right| + \left|\left\{x\in\mathbb{R}^d \setminus \cup_{\mathcal{B}} Q^* : \left|(Tf_2)(x)\right| > \frac{\lambda}{2}\right\}\right| \\
&\leq C\sum_{Q\in\mathcal{B}} |Q| + \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\cup Q^*} \left|(Tf_2)(x)\right|\mathrm{d}x \\
&\leq \frac{C}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\sum_{Q\in\mathcal{B}}\int_{\mathbb{R}^d\setminus Q^*} \left|(Tf_Q)(x)\right|\mathrm{d}x,
\end{aligned}$$

where we have used Chebyshev's inequality and the the property

$$\left|\bigcup_{Q\in\mathcal{B}} Q\right| < \frac{1}{\lambda}\|f\|_{L^1}$$

of $\mathcal{B}$. For any $x \in \mathbb{R}^d \setminus Q^*$ we have

$$(Tf_Q)(x) = \int_Q K(x-y) f_Q(y)\,\mathrm{d}y = \int_Q (K(x-y) - K(x-y_Q)) f_Q(y)\,\mathrm{d}y$$

since f_Q has mean value 0, where y_Q is the center of the cube Q. This expression allows us to exploit (K2):

$$\begin{aligned}\int_{\mathbb{R}^d\setminus Q^*} \left|(Tf_Q)(x)\right| \mathrm{d}x &\le \int_{\mathbb{R}^d\setminus Q^*}\int_Q \left|K(x-y) - K(x-y_Q)\right|\left|f_Q(y)\right| \mathrm{d}y\,\mathrm{d}x \\ &\le \int_Q \left|f_Q(y)\right| \mathrm{d}y \le 2\int_Q \left|f(y)\right| \mathrm{d}y.\end{aligned}$$

Note how the selection of Q^* allows us to apply (K2). Since the Q are disjoint, this finishes the proof. □

Now we use interpolation between the L^2 and weak-L^1 bounds to obtain L^p boundedness of singular integral operators.

Theorem 2.10. *Let T be a singular integral operator bounded on $L^2(\mathbb{R}^d)$ with kernel that satisfies (K1) and (K2). Then T can be extended to a bounded operator on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$ with an operator norm $\|T\|_{L^p\to L^p} \le C(p,d)B$.*

Proof. The statement for $1 < p < 2$ is a consequence of Marcinkiewicz interpolation, and the statement for $p = 2$ is assumed. The statement for $2 < p < \infty$ follows from a duality argument and is as follows. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then $\langle Tf, g\rangle = \overline{\langle f, T^*g\rangle}$ where T^* is a singular integral operator associated with kernel $K^*(x) = \overline{K(-x)}$. Clearly K^* satisfies (K1) and (K2), and a simple duality argument shows that T^* is bounded on L^2. Applying the same arguments to T^* as we have to T, we see that T^* is bounded on L^q for $1 < q \le 2$. Since $1 < p' \le 2$, we have

$$\begin{aligned}\|Tf\|_{L^p} &= \sup_{\|g\|_{L^{p'}}=1} \langle Tf, g\rangle = \sup_{\|g\|_{L^{p'}}=1} \langle f, T^*g\rangle \\ &\le \|f\|_{L^p} \sup_{\|g\|_{L^{p'}}=1} \|T^*g\|_{L^{p'}} \le CB\|f\|_{L^p}.\end{aligned}$$

□

3. The Mikhlin Multiplier Theorem

A Fourier multiplier is a function $m : \mathbb{R}^d \to \mathbb{C}$ that defines a corresponding operator T_m by $f \mapsto (m\hat{f})^\vee$ on the space of Schwartz functions. A natural problem is to determine conditions under which such operators can be extended as a bounded operator on L^p. For L^2, the problem is easy using Plancherel's theorem: the operator is bounded if and only if m is in $L^\infty(\mathbb{R}^d)$. The Mikhlin multiplier theorem gives sufficient conditions for such an operator to be bounded on L^p for $1 < p < \infty$; however, it is by no means a complete classification. To start, we must construct a partition of unity on a geometric scale.

Theorem 3.1. *There exists a function $\psi \in C^\infty(\mathbb{R}^d)$ such that supp$(\psi) \subset \mathbb{R}^d \setminus \{0\}$ is compact and such that*

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1.$$

for all $x \neq 0$. We call such a ψ a dyadic partition of unity, and define $\psi_j(x) = \psi(2^{-j}x)$. We will construct ψ to be radial, nonnegative, and so that only at most two terms in the sum are nonzero.

Proof. Let $\chi \in C^\infty(\mathbb{R}^d)$ be radial with $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) < 1$ for $1 < |x| < 2$, and $\chi(x) = 0$ for $|x| \geq 2$. Define $\psi(x) = \chi(x) - \chi(2x)$. It is immediately verified that ψ is radial and nonnegative. Moreover for any positive integer N we have

$$\sum_{j=-N}^{N} \psi(2^{-j}x) = \chi(2^{-N}x) - \chi(2^{N+1}x).$$

For N sufficiently large and $x \neq 0$, we have $\left|2^{-N}x\right| \leq 1$ and $\left|2^{N+1}x\right| \geq 2$, and so the sum is 1. This finishes the proof. $\square$

Theorem 3.2. (Mikhlin Multiplier Theorem) *Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfy*

$$\left|\partial^\gamma m(\xi)\right| \leq B|\xi|^{-|\gamma|}$$

for any multi-index γ with $|\gamma| \leq d+2$ and for all $\xi \neq 0$. Then there exist constants $C(d,p)$ for $1 < p < \infty$ such that

$$\left\|\left(m\hat{f}\right)^\vee\right\|_{L^p} \leq C(d,p)B\|f\|_{L^p}$$

for all $f \in \mathcal{S}$.

This theorem implies the L^p boundedness of a wide range of multipliers, e.g. the sgn function in one dimension and smooth functions of compact support.

Proof. With ψ a dyadic partition of unity, define $m_j(\xi) = \psi(2^{-j}\xi)m(\xi)$. Let $K_j = \overset{\vee}{m}_j$, and

$$K(x) = \sum_{j=-N}^{N} K_j(x).$$

for a fixed, large positive integer N. We will prove the estimates

$$\left|K(x)\right| \leq C(d)B|x|^{-d}, \quad \left|\nabla K(x)\right| \leq C(d)B|x|^{-d}. \tag{3.3}$$

hold uniformly in N. A calculation gives $\left\|\partial^\gamma m_j\right\|_{L^\infty} \leq CB2^{-j|\gamma|}$ for all multi-indices γ with $|\gamma| \leq d+2$. To see this, let $R > 0$ be such that supp$(\psi) \subset B(0,R)$. Then

$$\begin{aligned}
\left|\partial^\gamma\left(m(\xi)\psi(2^{-j}\xi)\right)\right| &\leq C\sum_{\alpha,\alpha<\gamma}\left|\partial^\alpha m(\xi)\right|\left|\partial^{\gamma-\alpha}\psi(2^{-j}\xi)\right| \\
&\leq CB\sum_{\alpha,\alpha<\gamma}|\xi|^{-|\alpha|}(1-\chi_{[B(0,2^jR)]})2^{-j(|\gamma|-|\alpha|)}\sup_{\beta,x}\left|\partial^\beta\psi(x)\right| \\
&\leq CB\sum_{\alpha,\alpha<\gamma}R^{-|\alpha|}2^{-j|\alpha|}2^{-j(|\gamma|-|\alpha|)} \leq CB2^{-j|\gamma|}.
\end{aligned}$$

This estimate and the compactness of $\operatorname{supp}(\psi)$ gives $\left\|\partial^\gamma m_j\right\|_{L^1} \le CB2^{-j|\gamma|}2^{jd}$. This result and another compactness argument yields

$$\left\|\partial^\gamma(\xi_i m_j)\right\|_{L_1} \le C\left\|\xi_i\partial^\gamma m_j\right\|_{L_1} + C\sum_{\alpha,|\alpha|=|\gamma|-1}\left\|\partial^\alpha m_j\right\|_{L^1} \le CB2^{-j(|\gamma|-1)}2^{jd}.$$

We have

$$\left\|x^\gamma \overset{\vee}{m}_j(x)\right\|_{L^\infty} = C\left\|(\partial^\gamma m_j)^\wedge\right\|_{L^\infty} \le C\left\|\partial^\gamma m_j\right\|_{L^1} \le CB2^{j(d-|\gamma|)}$$

and similarly

$$\left\|x^\gamma D\overset{\vee}{m}_j(x)\right\|_{L^\infty} \le CB2^{j(d+1-|\gamma|)}.$$

Note that these manipulations are justified by the compact support and differentiability of m_j. Since $|x|^k \le C(k,d)\sum_{\gamma,|\gamma|=k}|x^\gamma|$, we arrive at

$$\left|\overset{\vee}{m}_j(x)\right| \le CB2^{j(d-k)}|x|^{-k}, \quad \left|D\overset{\vee}{m}_j(x)\right| \le CB2^{j(d+1-k)}|x|^{-k} \tag{3.4}$$

for any $0 \le k \le d+2$ and all $j \in \mathbb{Z}$, $x \in \mathbb{R}^d \setminus \{0\}$. We will prove the first inequality in (3.3) by using the first inequality in (3.4) with $k = 0$ and $k = d+2$, and the same method can be used to prove the second inequality in (3.3) with the second inequality in (3.4). We have

$$\begin{aligned}
\left|K(x)\right| &\le \sum_{j=-\infty}^{\infty}\left|\overset{\vee}{m}_j(x)\right| \le \sum_{2^j\le|x|^{-1}}\left|\overset{\vee}{m}_j(x)\right| + \sum_{|x|^{-1}<2^j}\left|\overset{\vee}{m}_j(x)\right| \\
&\le CB\sum_{2^j\le|x|^{-1}}2^{jd} + CB\sum_{|x|^{-1}<2^j}2^{jd}(2^j|x|)^{-(d+2)} \\
&\le CB|x|^{-d} + CB|x|^2|x|^{-(d+2)} = CB|x|^{-d}.
\end{aligned}$$

This shows that K satisfies (K1), independent on N. A similar proof yields $\left|\nabla K(x)\right| < CB|x|^{-d-1}$, which implies K satisfies (K2). Note

$$\left\|(m\hat{f})^\vee\right\|_{L^2} = \left\|m\hat{f}\right\|_{L^2} \le \|m\|_{L^\infty}\left\|\hat{f}\right\|_{L^2} = \|m\|_{L^\infty}\|f\|_{L^2},$$

and since $\|m\|_{L^\infty} \le B$ by hypothesis, we know that m is bounded as a Fourier multiplier on L^2. Sending N to infinity, the resulting operator satisfies the requirements of a Calderón-Zygmund operator with L^2 boundedness in place of (K3). Therefore, the operator is bounded on L^p for $1 < p < \infty$, as desired. □

We remark here that while the Mikhlin multiplier theorem provides useful sufficient conditions for L^p boundedness of Fourier multipliers, it is not discerning enough for a satisfactory theory; it does not distinguish between p so long as $1 < p < \infty$.

An interesting consequence of the Mikhlin multiplier theorem is a class of estimates important for elliptic partial differential equations.

Corollary 3.5. *A Fourier multiplier m that is homogenous of degree 0 and smooth everywhere except at 0 defines an operator bounded on L^p. As a consequence, we have the following Schauder estimate:*

$$\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L^p} \le C(p,d)\|\triangle u\|_{L^p} \tag{3.6}$$

for $u \in \mathcal{S}(\mathbb{R}^d)$, for indices i, j with $1 \leq i, j \leq d$, for $1 < p < \infty$, and for $C(p,d)$ a constant depending only on d and p.

Proof. By writing a difference quotient, we see that the directional derivative of m at x is maximized when taken along the direction perpendicular to x. By homogeneity, we see that this derivative is equal to the directional derivative of m at $x/|x|$ scaled by $1/|x|$. By continuing writing difference quotients, we see that an nth order partial derivative of m at x is equal to the nth order partial derivative of m at $x/|x|$ scaled by $1/|x|^n$. By smoothness of m and compactness of the unit circle, we can then bound all partial derivatives of order less than or equal to $d+2$ globally. Thus m satisfies the hypotheses of theorem 3.2, and is a Fourier multiplier bounded from L^p to L^p for $1 < p < \infty$.

The Schauder estimate (3.6) follows from the formula

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^{\wedge}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}(\triangle u)^{\wedge}(\xi)$$

and the fact that $m(\xi) = \xi_i\xi_j/|\xi|^2$ is homogenous of degree 0. $\square$

4. Littlewood-Paley Theory

In this section, we develop the basic ideas of Littlewood-Paley theory. Define $P_j f = (\psi_j \hat{f})^{\vee}$ for $j \in \mathbb{Z}$ and $f \in \mathcal{S}(\mathbb{R}^d)$, where the ψ_j are the functions defined in 3.1. Define the Littlewood-Paley square function Sf by

$$Sf = \left(\sum_{j=-\infty}^{\infty} |P_j f|^2\right)^{1/2}.$$

Using Plancherel's theorem, we have

$$\|Sf\|_{L^2}^2 = \sum_{j=-\infty}^{\infty} \|P_j f\|_{L^2}^2 = \sum_{j=-\infty}^{\infty} \left\|\psi_j \hat{f}\right\|_{L^2}^2 = \int_{\mathbb{R}^d} \left(\sum_{j=-\infty}^{\infty} |\psi_j(x)|^2\right) \left|\hat{f}(x)\right|^2 \mathrm{d}x.$$

From this equality, it is easy to see that $\|Sf\|_{L^2} \leq \|f\|_{L^2}$. By using the fact that at most two terms of $\sum_j |\psi_j(x)|^2$ are nonzero, it can be shown that in fact $\|Sf\|_{L^2} \geq C^{-1}\|f\|_{L^2}$ for an absolute constant C. It is natural to ask: does a similar inequality hold for p other than 2? The goal of this section is to prove such an inequality for $1 < p < \infty$.

Define $\rho_N = \{r = (r_j)_{j=1}^N : r_j = \pm 1\}$. For any finite set A let $\#A$ be the number of elements in A. The following two lemmas come from probability theory, but here will be presented as analytical facts in the interest of being self-contained.

Lemma 4.1. *For any $(a_j)_{j=1}^N \subset \mathbb{C}$ with $\sum_{j=1}^N |a_j| = 1$, we have*

$$\#\left\{r \in \rho_N : \left|\sum_{j=1}^N r_j a_j\right| > \lambda\right\} \leq 4e^{-\lambda^2/2} \#\rho_N \tag{4.2}$$

for all $\lambda > 0$.

Proof. Set $S_N(r) = \sum_{j=1}^N r_j a_j$. Make first the assumption that $(a_j)_{j=1}^N \subset \mathbb{R}$. The more general case will follow from decomposition into real and imaginary parts. It is easy to check via an expansion that

$$\frac{1}{\#\rho_N} \sum_{r \in \rho_N} e^{\lambda S_N(r)} = \prod_{j=1}^N \frac{1}{2} \sum_{r_j = \pm 1} e^{\lambda r_j a_j} = \prod_{j=1}^N \cosh(\lambda a_j).$$

Using the bound $\cosh(x) \leq \exp(x^2/2)$ from calculus, one has

$$\frac{1}{\#\rho_N} \sum_{r \in \rho_N} e^{\lambda S_N(r)} \leq \exp\left(\frac{1}{2}\lambda^2 \sum_{j=1}^N a_j^2\right) = e^{\lambda^2/2}.$$

Combining this with the Chebyshev inequality

$$\sum_{r \in \rho_N} e^{\lambda S_N(r)} \geq e^{\lambda^2} \# \left\{ r \in \rho_N : e^{\lambda S_N(r)} > e^{\lambda^2} \right\} = e^{\lambda^2} \# \left\{ r \in \rho_N : S_N(r) > \lambda \right\}$$

gives

$$\# \left\{ r \in \rho_N : S_N(r) > \lambda \right\} \leq e^{-\lambda^2} e^{\lambda^2/2} \#\rho_N = e^{-\lambda^2/2} \#\rho_N.$$

A similar argument gives

$$\# \left\{ r \in \rho_N : S_N(r) < -\lambda \right\} \leq e^{-\lambda^2/2} \#\rho_N$$

whence the lemma holds for our restricted (a_j). By decomposing the (a_j) into real and imaginary parts (b_j) and (c_j) and noting

$$\left| \sum_{j=1}^N r_j a_j \right|^2 = \left| \sum_{j=1}^N r_j b_j \right|^2 + \left| \sum_{j=1}^N r_j c_j \right|^2$$

we have reduced the lemma to one of the previous cases, losing a factor of two in the process. □

The following result is known as Klinchine's inequality.

Lemma 4.3. *Let $1 < p < \infty$. Then, for any $r \in \rho_N$ and $(a_j)_{j=1}^N \subset \mathbb{C}$, one has*

$$\frac{1}{C(p)} \left(\sum_{j=1}^N \left| a_j \right|^2 \right)^{p/2} \leq \frac{1}{\#\rho_N} \sum_{r \in \rho_N} \left| \sum_{j=1}^N r_j a_j \right|^p \leq C(p) \left(\sum_{j=1}^N \left| a_j \right|^2 \right)^{p/2} \tag{4.4}$$

for some constant $C(p)$.

Proof. Let $S_N(r) = \sum_{j=1}^N r_j a_j$ as before. By dividing each term of (a_j) by $\sum_{j=1}^N |a_j|$, we may assume $\sum_{j=1}^N |a_j| = 1$. The upper bound will be obtained through the estimate achieved in Lemma 4.1. We have

$$\begin{aligned} \frac{1}{\#\rho_N} \sum_{r \in \rho_N} \left| S_N(r) \right|^p &= \frac{1}{\#\rho_N} \int_0^\infty \#\{ r \in \rho_N : \left| S_N(r) \right| > \lambda \} p \lambda^{p-1} \, \mathrm{d}\lambda \\ &\leq \int_0^\infty 4 e^{-\lambda^2/2} p \lambda^{p-1} \, \mathrm{d}\lambda = C(p) < \infty. \end{aligned}$$

Note that

$$\sum_{r \in \rho_N} \left| S_N(r) \right|^2 = \sum_{r \in \rho_N} \left(\sum_{j=1}^N a_j^2 r_j^2 + \sum_{n \neq m} c_{n,m} r_n r_m \right)$$

where $c_{n,m}$ depends on the sequence (a_j). But noting that $r_n r_m$ is equally often $+1$ and -1, we see that $\sum_{r\in\rho_N}\left|S_N(r)\right|^2 = \#\rho_N \sum\left|a_j\right|^2 = \#\rho_N$. This shows that equality holds in the lemma for $p=2$. We leverage this result and the lower bound to finish the lemma. By applying Holder's inequality, we have

$$1 = \frac{1}{\#\rho_N}\sum_{r\in\rho_N}\left|S_N(r)\right|^2 \leq \left(\frac{1}{\#\rho_N}\sum_{r\in\rho_N}\left|S_N(r)\right|^{p'}\right)^{1/p'}\left(\frac{1}{\#\rho_N}\sum_{r\in\rho_N}\left|S_N(r)\right|^{p}\right)^{1/p}$$
$$\leq C(p')^{1/p'}\left(\frac{1}{\#\rho_N}\sum_{r\in\rho_N}\left|S_N(r)\right|^{p}\right)^{1/p},$$

and the proof is finished. □

Now that we have established the Mikhlin Multiplier theorem and Klinchine's inequality, we are ready to prove the deep Littlewood-Paley square function estimate.

Theorem 4.5. *For $1<p<\infty$, the Littlewood-Paley square function Sf has an L^p norm comparable to the L^p norm of f. That is*

$$C(p,d)^{-1}\|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C(p,d)\|f\|_{L^p}$$

for some constant $C(p,d)$.

Proof. Define the Fourier multipliers

$$m_{N,r}(\xi) = \sum_{j=-N}^{N} r_j\psi_j(\xi)$$

for every $r\in\rho_N$ and every N. We claim that the hypotheses of the Mikhlin multiplier theorem are satisfied uniformly in N and r. We have

$$\left|\partial^\gamma m_{N,r}(\xi)\right| \leq \sum_{j=-N}^{N}\left|\partial^\gamma\psi_j(\xi)\right| = \sum_{j=-N^N} 2^{-j|\gamma|}\left|(\partial^\gamma\psi)(2^{-j}\xi)\right|.$$

for any multi-index γ and $\xi\neq 0$. Note that $2^{-j|\gamma|}$ is within a factor of two of $|\xi|^{-|\gamma|}$ whenever $(\partial^\gamma\psi)(2^{-j}\xi)$ is nonzero. Moreover, only an absolutely bounded number of terms of the final sum (here exactly two) are nonzero at any one time. This allows us to assert

$$\left|\partial^\gamma m_{N,r}(\xi)\right| \leq C\sum_{j=-N^N}|\xi|^{-|\gamma|}\left|(\partial^\gamma\psi)(2^{-j}\xi)\right| \leq C|\xi|^{-|\gamma|}.$$

Note that this inequality works trivially for $\xi=0$. We now use Lemma 4.3 with $a_j=(P_jf)(x)$ to say

$$\left(\sum_{j=-N}^{N}\left|(P_jf)(x)\right|^2\right)^{p/2} \leq \frac{C}{\#\rho_N}\sum_{r\in\rho_N}\left|\sum_{j=-N}^{N} r_j(P_jf)(x)\right|^p$$

for all x. If we now integrate both sides, take the lim sup in N, and apply the Mikhlin multiplier theorem, we have

$$\int_{\mathbb{R}^d} \left|(Sf)(x)\right|^p \mathrm{d}x \leq \limsup_{N\to\infty} \frac{C}{\#\rho_N} \sum_{r\in\rho_N} \int_{\mathbb{R}^d} \left| \sum_{j=-N}^{N} r_j (P_j f)(x) \right|^p \mathrm{d}x \leq C\|f\|_p^p .$$

Duality is used to prove the lower bound. Let $\tilde{\psi}$ be a function that is 1 on $\operatorname{supp}(\psi)$, compactly supported, and has $0 \notin \operatorname{supp}(\tilde{\psi})$. Define $\tilde{P}_j$ with $\tilde{\psi}$ in place of ψ, and similarly define $\tilde{S}$. Then clearly $\tilde{P}_j P_j = P_j$. Let $f, g \in \mathcal{S}$ be arbitrary. Applying Plancherel's theorem twice, we have

$$\begin{aligned} \left|\langle f, g\rangle\right| = \left| \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}}(\xi)\,\mathrm{d}\xi \right| &= \left| \int_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} \hat{f}(\xi)\psi_j(\xi)\overline{\hat{g}(\xi)\tilde{\psi}_j(\xi)}\,\mathrm{d}\xi \right| \\ &= \left| \sum_{j=-\infty}^{\infty} \left\langle P_j f, \tilde{P}_j g \right\rangle \right| . \end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality and Holder's inequality, we have

$$\begin{aligned} \left|\langle f, g\rangle\right| &\leq \int_{\mathbb{R}^d} \left(\sum_{j=-\infty}^{\infty} \left|(P_j f)(x)\right|^2 \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} \left|(\tilde{P}_k g)(x)\right|^2 \right)^{1/2} \\ &\leq \|Sf\|_p \left\|\tilde{S}g\right\|_{p'} \leq C\|Sf\|_p \|g\|_{p'} , \end{aligned}$$

where the upper bound on $\left\|\tilde{S}g\right\|_{p'}$ comes from the same argument we used for the upper bound of $\|Sf\|_p$ applied to $\tilde{S}$. Noting that $\|f\|_p = \sup_{\|g\|_{p'}=1} \left|\langle f, g\rangle\right|$, we see that this proves the lower bound. □

The failure of this theorem at $p = 1$ can be seen through the action of S on the Dirac delta distribution; in particular $S\delta_0 \simeq 1/|x|^d$. As usual, one must argue by approximating δ_0 with an approximate identity and using a limiting process. The failure at $p = \infty$ is a consequence of failure at $p = 1$ and duality.

5. The Ball Multiplier Theorem

It is a remarkable fact that the operator S given by

$$Sf(x) = \int_{[\![\xi|<1]\!]} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi, \tag{5.1}$$

initially defined on $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, is not bounded with respect to the L^p norm for $p \neq 2$ and $d > 1$ (boundedness holds for $d = 1$ due to the boundedness of the Hilbert Transform and for $p = 2$ due to Plancherel's theorem). The operator S corresponds to the Fourier multiplier $\chi_{B(0,1)}$, i.e. the ball multiplier. The proof of this fact relies on the existence of so-called Kakeya sets of arbitrarily small measure. These are subsets of the plane that contain a unit line segment in every direction. We do not construct such sets here, and refer the reader to [2, p.433] to see a detailed construction. Instead, we state the lemma we will need.

Lemma 5.2. *For any $\epsilon > 0$, there exists an integer $N \geq 1$ and 2^N rectangles of side lengths 1 and 2^{-N}, denoted $R_1, \dots, R_{2^N}$, such that*

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \epsilon, \tag{5.3}$$

and the translates of each R_j by 2 units in the positive direction along its longer side, denoted R'_j, are mutually disjoint. Therefore

$$\left| \bigcup_{j=1}^{2^N} R'_j \right| = 1. \tag{5.4}$$

Here $|\cdot|$ denotes Lebesgue measure.

Note that the rectangles, which constitute a Kakeya set, have a high degree of overlap, yet point in many different directions. We proceed in the proof of the ball multiplier theorem by contradiction. Along with existence of small Kakeya sets, the following lemma plays a key role.

Lemma 5.5. *Suppose that the following bound holds for all $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some p with $1 \leq p \leq \infty$:*

$$\|Sf\|_{L^p} \leq A_p \|f\|_{L^p}\,. \tag{5.6}$$

Let $u_1, \dots, u_M$ be unit vectors in $\mathbb{R}^d$ for some fixed integer M. Then

$$\left\| \left(\sum_{j=1}^{M} \left| H_{u_j}(f_j) \right|^2 \right)^{1/2} \right\|_{L^p} \leq A_p \left\| \left(\sum_{j=1}^{M} \left| f_j \right|^2 \right)^{1/2} \right\|_{L^p} \tag{5.7}$$

where

$$H_u(f)(x) = \int_{[\xi \cdot u > 0]} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi.$$

Proof. Let $f = (f_1, \dots, f_M)$ and let $T(f) = (Tf_1, \dots, Tf_M)$ for any linear operator T. Define the family of operators T_ω by

$$T_\omega(f) = \sum_{j=1}^{M} \overline{\omega}_j T(f_j) \tag{5.8}$$

where $\omega = (\omega_j)_{j=1}^M \in \mathbb{C}^m$ satisfies $|\omega| = 1$. Similarly define

$$f_\omega = \sum_{j=1}^{M} \overline{\omega}_j f_j.$$

Suppose that we have the bound (for $p < \infty$)

$$\int_{\mathbb{R}^d} \left| T_\omega f(x) \right|^p \mathrm{d}x \leq A_p^p \int_{\mathbb{R}^d} \left| f_\omega(x) \right|^p \mathrm{d}x. \tag{5.9}$$

Define $\phi(\omega, u) = \left\langle u/|u|\,, \omega \right\rangle = (u/|u|) \cdot \overline{\omega}$ for $u \neq 0$ and $\phi(\omega, 0) = 0$. Clearly we have

$$\left| T_\omega f(x) \right| = \left| Tf(x) \right| \left| \phi(\omega, Tf(x)) \right| = \left(\sum_{j=1}^{M} \left| Tf_j(x) \right|^2 \right)^{1/2} \left| \phi(\omega, Tf(x)) \right|. \tag{5.10}$$

Define

$$\gamma(p,f,x)=\int_{[\![\omega|=1]}\left|\phi(\omega,Tf(x))\right|^p \,\mathrm{d}\omega.$$

If $Tf(x)\neq 0$ then there exists a rotation Q such that

$$Q(Tf(x)/\left|Tf(x)\right|)=(1,0,\ldots,0)=e_1.$$

Since rotations preserve the Hermitian inner product and the resulting change of variables preserves $\mathrm{d}\omega$, we have

$$\gamma(p,f,x)=\int_{[\![\omega|=1]}\left|\phi(Q\omega,e_1)\right|^p \,\mathrm{d}\omega=\int_{[\![\omega|=1]}\left|\phi(\omega,e_1)\right|^p \,\mathrm{d}\omega,$$

whence $\gamma(p,f,x)=\gamma_p$ is independent of f and x and is nonzero. The case $Tf(x)=0$ is immaterial in the proof.

We integrate (5.9) over $\{\omega : |\omega|=1\}$ before integrating in x. Using the independence of γ_p from x and $\gamma_p>0$, we conclude that

$$\left\|\left(\sum_{j=1}^M\left|T(f_j)\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}\leq A_p\left\|\left(\sum_{j=1}^M\left|f_j\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \tag{5.11}$$

Now let $p=\infty$. If we assume in place of (5.9) the inequality

$$\|T_\omega f\|_{L^\infty}\leq A_\infty\|f_\omega\|_{L^\infty}\,, \tag{5.12}$$

then taking $\operatorname{ess\,sup}_{|\omega|=1}$ of both sides inside of the L^∞ norm and noting $\operatorname{ess\,sup}\left|\phi(\omega,u)\right|=1$ for all u gives (5.11) for $p=\infty$.

We now show that (5.9) and (5.12) and therefore (5.11) hold for operators of interest. Since $T_\omega(f)=T(f_\omega)$, any operator satisfying the bound (5.6) is such an operator. For any set $E\subset\mathbb{R}^d$ define

$$S_Ef(x)=\int_E\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,\mathrm{d}\xi.$$

The hypothesis (5.6) implies an identical bound for $S_{B(0,R)}$, where $B(0,R)$ is the ball of radius R centered at the origin, for any $R>0$. Indeed, letting $\delta_R(g)(x)=g(x/R)$, we have

$$\left\|S_{B(0,R)}f\right\|_{L^p}=\left\|\delta_R^{-1}S\delta_Rf\right\|_{L^p}=R^{-d}\|S\delta_Rf\|_{L^p}\leq R^{-d}A_p\|\delta_Rf\|_{L^p}=A_p\|f\|_{L^p}\,.$$

Let $u\in\mathbb{R}^d$ be a unit vector. It is easy to verify

$$S_{B(Ru,R)}f(x)=e^{2\pi iRu\cdot x}S_{B(0,r)}(e^{-2\pi iRu\cdot x}f),$$

in view of which

$$\left\|\left(\sum_{j=1}^M\left|S_{B(Ru,R)}(f_j)\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}\leq A_p\left\|\left(\sum_{j=1}^M\left|f_j\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}. \tag{5.13}$$

As $R\to\infty$, the set $B(Ru,R)$ increases to the half space $\{\xi:\xi\cdot u>0\}$. By monotone convergence, we have $S_{B(Ru,R)}$ converges to H_u in L^2, and so a subsequence converges almost everywhere, whence (5.13) proves the lemma. $\square$

We now formally state and prove the ball multiplier theorem.

Theorem 5.14. *The operator S cannot be extended to a bounded operator on L^p.*

Proof. Consider the operator $S_{\mathbb{R}^+}$ defined by

$$(S_{\mathbb{R}^+}f)(x) = \int_0^\infty \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi$$

for $f \in L^2(\mathbb{R}^1)$. Put $f = \chi_{(-1/2,1/2)}$. We have

$$(S_{\mathbb{R}^+}f)(x) = \lim_{\varepsilon\to 0^+}\int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\varepsilon)\xi}\,\mathrm{d}\xi.$$

We also have

$$\begin{aligned}\int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\varepsilon)\xi}\,\mathrm{d}\xi &= \int_{-\infty}^\infty \left(\int_0^\infty e^{-2\pi i y\xi}e^{2\pi i(x+i\varepsilon)\xi}\,\mathrm{d}\xi\right) f(y)\,\mathrm{d}y\\ &= \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(y)}{(y-x-i\varepsilon)}\,\mathrm{d}y\end{aligned}$$

where the change in integration order is justified by

$$\int_0^\infty\int_{-\infty}^\infty \left|f(y)\right| e^{-2\pi\epsilon\xi}\,\mathrm{d}y\,\mathrm{d}\xi = \int_0^\infty e^{-2\pi\epsilon\xi}\,\mathrm{d}\xi < \infty.$$

Let $|x| > 1/2$ and $0 < \varepsilon < 1/2$. By the intermediate value theorem, we write

$$\int_{-1/2}^{1/2}\frac{1}{y-x-i\epsilon}\,\mathrm{d}y = \frac{1}{c-x-i\epsilon}$$

for some c with $-1/2 \leq |c| \leq 1/2$, from which it follows

$$\left|(S_{\mathbb{R}^+}f)(x)\right| \geq \frac{C}{|x|}, \text{ for } |x| > 1/2. \tag{5.15}$$

We will use this bound along with the Kakeya construction to show a contradiction in (5.7).

Set $p < 2$. Let f be the indicator function for the ball of radius 3 in $\mathbb{R}^{d-2}$; if $n = 2$ let f be the constant 1. Let $R_1, \ldots, R_{2^N}$ be the rectangles constructed by lemma 5.2. Note that if $F(x_1, x_2) = f_1(x_1)f_2(x_2)$ and e_1 is the unit vector in the x_1 direction, we have that $(H_{e_1}F)(x_1, x_2) = (S_{\mathbb{R}^+}f_1)(x_1)f_2(x_2)$. Let $u_{j,1}$ and $u_{j,2}$ be unit vectors in the direction of the longest and shortest sides of R_j, respectively, and let $u_{j,3}$ stand in for the remaining $n-2$ coordinates. Define $f_j(x_1, x_2, x_3) = \chi_{R_j}(x_1, x_2)f(x_3)$ for $j = 1, \ldots, 2^N$, where x_1, x_2, x_3 are coordinates with respect to $u_{j,1}, u_{j,2}, u_{j,3}$. We see

$$\begin{aligned}\left|(H_{u_{j,1}}f_j)(x_1,x_2,x_3)\right| &= \left|(S_{\mathbb{R}^+}\chi_{(-1/2,1/2)})(x_1)\chi_{(-2^{-N-1},2^{-N-1})}(x_2)f(x_3)\right|\\ &\geq \frac{C}{|x_1|}\left|\chi_{(-2^{-N-1},2^{-N-1})}(x_2)f(x_3)\right|\\ &\geq c'\chi_{R'_j}(x_1,x_2)\left|f(x_3)\right|\end{aligned} \tag{5.16}$$

for some $c' > 0$ ($c' = 2C/5$ works), where R'_j is the translate of R_j as defined in lemma 5.2. Setting $M = 2^N$ and $u_j = u_{j,1}$, we compute that the left side of (5.7) is larger than $c'v(B_3)$, where $v(B_3)$ is defined to be the volume of the ball of radius 3 in $\mathbb{R}^{d-2}$ if $d > 2$ and is defined to be 1 if $d = 2$. This is because $\left|\cup_j R'_j\right| = 1$

and the R_j' are disjoint. Applying Holder's inequality with dual exponents $2/p$ and $q = 1/(1-p/2)$, we conclude that the right side of (5.7) is smaller than

$$A_p \left[\int_{\mathbb{R}^d} \left(\sum_{j=1}^{2^N} \left| \chi_{R_j}(x_1, x_2) f(x_3) \right|^2 \right) \mathrm{d}x \right] \left(\int_{\cup_j R_j \times B_3} \mathrm{d}x \right)^{1/q} . \tag{5.17}$$

Note that, by our choice of $p < 2$ and since $\left| \cup_j R_j \right|$ can be made arbitrarily small via lemma 5.2, the rightmost factor of (5.17) can be made arbitrarily small. And

$$\int_{\mathbb{R}^d} \left(\sum_{j=1}^{2^N} \left| \chi_{R_j}(x_1, x_2) f(x_3) \right|^2 \right) \mathrm{d}x = v(B_3) \sum_{j=1}^{2^N} \left| R_j \right| = v(B_3),$$

whence (5.17) can be made smaller than $c' v(B_3)$, contradicting (5.7). Since the L^p boundedness of the ball multiplier, (5.6), implies (5.7), we have the theorem for $p < 2$ and all $d > 1$.

We now extend the result to $p > 2$ by duality. Suppose that S is bounded as an operator for some $p > 2$. Let $p' < 2$ denote the dual exponent of p. Then, noting that $\langle Sf, g \rangle = \langle f, Sg \rangle$, we have

$$\|Sg\|_{L^{p'}} = \sup_{\|f\|_{L^p} = 1} \langle f, Sg \rangle = \sup_{\|f\|_{L^p} = 1} \langle Sf, g \rangle \leq \sup_{\|f\|_{L^p} = A_p} \langle f, g \rangle = A_p \|g\|_{L^{p'}} ,$$

contradicting the preceding result. Therefore S cannot be extended to a bounded operator on $L^p(\mathbb{R}^d)$ for $p \neq 2$ and $d > 1$. □

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