INTRODUCTION TO THE CONVERGENCE OF SEQUENCES

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Abstract. In this paper, we discuss the basic ideas involved in sequences and convergence. We start by defining sequences and follow by explaining convergence and divergence, bounded sequences, continuity, and subsequences. Relevant theorems, such as the Bolzano-Weierstrass theorem, will be given and we will apply each concept to a variety of exercises.

Contents

1. Introduction to Sequences

Definition 1.1. A *sequence* is a function whose domain is $\mathbf{N}$ and whose codomain is $\mathbf{R}$. Given a function $f\colon \mathbf{N} \to \mathbf{R}$, $f(n)$ is the nth term in the sequence.

Example 1.2. The first example of a sequence is $x_n = \frac{1}{n}$. In this case, our function f is defined as f$(n) = \frac{1}{n}$. As a listed sequence of numbers, this would look like the following:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} ...\right) \tag{1.3}$$

Another example of a sequence is $x_n = 5^n$, which would look like the following:

$$(5, 25, 125, 625...) \tag{1.4}$$

We know these are both valid examples of sequences because they are *infinite* lists of real numbers and hence can be regarded as functions with domain $\mathbf{N}$.

Example 1.5. The following would not be an examples of sequences:

$$(1, 2, 3) \tag{1.6}$$

$$(500, 200, 550, 10000) \tag{1.7}$$

Date: July 12, 2015.

We know that these are not examples of sequences because they are *finite* lists of real numbers.

2. Limit of a Sequence

A limit describes how a sequence x_n behaves "eventually" as n gets very large, in a sense that we make explicit below.

Definition 2.1. A sequence of real numbers *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that for all $n \geq N$, $|a_n$ - $a| < \epsilon$. We call such an a the limit of the sequence and write $\lim_{n\to\infty} a_n = a$.

Proposition 1. *The sequence $\frac{1}{n}$ converges to zero.*

Proof. Let $\epsilon > 0$. We choose $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. Such a choice is always possible by the Archimedean property. To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then, $n \geq N$ implies $n > \frac{1}{\epsilon}$ which is equal to $\frac{1}{n} = \mid \frac{1}{n} - 0 \mid < \epsilon$, proving that $\frac{1}{n}$ converges to zero by the definition of convergence. □

Proposition 2. *An example of a sequence that does not converge is the following:*

$$(1, -1, 1, -1, ...) \tag{2.2}$$

If a sequence does not converge, it is said to *diverge*, which we will explain later in the paper, along with the explanation of why the above sequence does not converge.

Proposition 3. *If $x_n \leq y_n \leq z_n$ for all $n \in \boldsymbol{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$, then $\lim_{n\to\infty} y_n = l$ too.*

Proof. Let $\epsilon > 0$. We want to show there exists an N such that for all $n > N$, $\mid y_n - l \mid < \epsilon$. We know that $x_n \to l$. Therefore, there exists an N_1 such that for all $n > N_1$, $\mid x_n - l \mid < \epsilon$. Also, we know that $z_n \to l$. Therefore, there exists an N_2 such that for all $n > N_2$, $\mid z_n - l \mid < \epsilon$. Let $N = \max(N_1, N_2)$ and $n > N$. Then, $n > N_1$ so $\mid x_n - l \mid < \epsilon$. Also, $n > N_2$ so $\mid z_n - l \mid < \epsilon$. We want to show that $\mid y_n - l \mid < \epsilon$. This is equivalent to showing that both $y_n - l < \epsilon$ and $l - y_n < \epsilon$. We know that $y_n \leq z_n$, so $y_n - l \leq z_n - l < \epsilon$. Also, we know that $y_n \geq x_n$, so $l - y_n \leq l - x_n < \epsilon$. □

Theorem 2.3 (Algebraic Limit Theorem)**.** *Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,*
(i) $\lim_{n\to\infty} ca_n = ca$ for all $c \in \boldsymbol{R}$
(ii) $\lim_{n\to\infty}(a_n + b_n) = a+b$
(iii) $\lim_{n\to\infty}(a_n b_n) = ab$
(iv) $\lim_{n\to\infty}(a_n/b_n) = a/b$ provided $b \neq 0$

Example 2.4. If $(x_n) \to 2$, then $((2x_n$ - $1)/3) \to 1$.

Proof. First, we will start with the information given in the example:

$$x_n \to 2.$$

Next, we simply use the fact that $(\frac{4}{3})(\frac{3}{2}) = 2$.

$$x_n \to \left(\frac{4}{3}\right)\left(\frac{3}{2}\right) \tag{2.5}$$

Now, let $a_n = x_n$, and let a = $(\frac{4}{3})(\frac{3}{2})$, and let $c = (\frac{2}{3})$. From the Algebraic Limit Theorem, we know that $ca_n \to ca$. Then, $(\frac{2}{3})(x_n) \to (\frac{2}{3})(\frac{4}{3})(\frac{3}{2})$, which is equal to the following:

$$\frac{2x_n}{3} \to \frac{4}{3} \tag{2.6}$$

The next step follows from the fact that $\frac{4}{3} = 1 + \frac{1}{3}$.

$$\frac{2x_n}{3} \to 1 + \frac{1}{3} \tag{2.7}$$

Let $\frac{2x_n}{3} = a_n$, let $(1+\frac{1}{3}) = a$, let $b_n = (\frac{-1}{3}, \frac{-1}{3}, ...)$, and let $b = \frac{-1}{3}$. Then, by the Algebraic Limit Theorem, we know that $a_n + b_n \to a + b$. Therefore, we know that $\frac{2x_n}{3} + \frac{-1}{3} \to (1+\frac{1}{3}) + \frac{-1}{3}$, which is equal to the following:

$$\frac{2x_n}{3} - \frac{1}{3} \to 1 \tag{2.8}$$

This last step follows because $\frac{2x_n}{3}$ - $\frac{1}{3} = \frac{2x_n - 1}{3}$.

$$\frac{2x_n - 1}{3} \to 1 \tag{2.9}$$

Therefore, using the Algebraic Limit Theorem, we have shown that if $(x_n) \to 2$, then $((2x_n$ - $1)/3) \to 1$.

□

Example 2.10. The following sequence converges to the proposed limit

$$lim \left(\frac{2n+1}{5n+4}\right) = \frac{2}{5} \tag{2.11}$$

Proof. Let $\frac{2n}{5n+4}$ be a_n, let $\frac{1}{5n+4}$ be b_n and let $\frac{2n+1}{5n+4}$ be c_n, and $c_n = a_n + b_n$. By Theorem 2.3, we know that $\lim(c_n) = \lim\ (a_n+b_n) = \lim(a_n)+\lim(b_n)$. We must therefore determine what $\lim(a_n)$ and $\lim(b_n)$ are.

First, we will show that $\lim(\frac{1}{5n+4}) = 0$. Let $\epsilon > 0$. By the Archimedean principle, there exists an $N \in \mathbf{N}$ such that $N > 1/\epsilon$. Then, for $n > N$, $\frac{1}{5n+4} < \frac{1}{5N+4} < 1/N < \epsilon$. Therefore, the limit of $\frac{1}{5n+4}$ is zero.

Then, because $\lim(c_n) = \lim(a_n+b_n)$, $\lim(c_n) = \lim(a_n + 0) = \lim(a_n)$. We will therefore find the limit of a_n in order to prove $\lim(\frac{2n+1}{5n+4}) = \frac{2}{5}$.

We now want to show that $\lim(\frac{2n}{5n+4}) = \frac{2}{5}$. Let $\epsilon > 0$. By the Archimedean Principle, there exists an N such that $1/\epsilon <$ N. Let $n > N$. We then want to show the following:

$$\left|\frac{2n}{5n+4} - \frac{2}{5}\right| < \epsilon \tag{2.12}$$

Then,

$$\left|\frac{2n}{5n+4} - \frac{2}{5}\right| = \left|\frac{-8}{5(5n+4)}\right| \tag{2.13}$$

We have to check the following:

$$\frac{-8}{5(5n+4)} < \epsilon \tag{2.14}$$

$$\frac{8}{5(5n+4)} < \epsilon \tag{2.15}$$

We know that the inequality $\frac{-8}{5(5n+4)} < \epsilon$ is true for every value of n because $n > N > 1/\epsilon$ and ϵ. Therefore we only need to show that the inequality $\frac{8}{5(5n+4)} < \epsilon$ is true. Using the fact that $N > 1/\epsilon$, we can say the following:

$$\frac{8}{5(5n+4)} < \frac{8}{5(5(1/\epsilon)+4)} = \frac{8\epsilon}{25+20\epsilon} \tag{2.16}$$

Then, $\frac{8\epsilon}{25+20\epsilon} < \frac{8\epsilon}{25} < \epsilon$. Therefore, $\frac{8}{5(5n+4)} < \epsilon$.

□

Example 2.17. Let $x_n \geq 0$. If $(x_n) \to 0$, then $(\sqrt{x_n}) \to 0$.

Proof. First, we have to prove that $\lim(\sqrt{x_n})$ exists. We know that x_n is decreasing but is greater than or equal to 0 for all values of n. The square root of a positive number is also positive. Therefore, $\sqrt{x_n} \geq 0$. Also, note that if $0 < a < b$, then $0<\sqrt{a}<\sqrt{b}$. So if x_n is decreasing, then so is $\sqrt{x_n}$. Therefore, $\lim(\sqrt{x_n})$ exists.

Next, we must prove that $(\sqrt{x_n}) \to 0$. Let $\lim(x_n) = \lim((\sqrt{x_n})(\sqrt{x_n})) = 0$. By the Algebraic Limit Theorem, we know that if $\lim(a_n) = a$ and $\lim(b_n) = b$ then $\lim((a_n)(b_n)) = ab$. By this theorem, $\lim((\sqrt{x_n})(\sqrt{x_n})) = \lim(\sqrt{x_n})\lim(\sqrt{x_n}) = 0$. Thus, $(\lim(\sqrt{x_n}))^2=0$. This implies that $\lim(\sqrt{x_n})=0$.

□

3. Divergence and Bounded Sequences

Definition 3.1. A sequence that does not have a limit or in other words, does not converge, is said to be *divergent*.

Example 3.2. Recall proposition 2, which says that the following sequence does not converge:

$$(1, -1, 1, -1...) \tag{3.3}$$

Later in this paper, we will give a concise proof of this fact. Contrast this with the following sequence, which we have seen

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}...\right) \tag{3.4}$$

This converges to zero, as we proved earlier in this paper. However, these sequences do have something in common. They are both *bounded*.

Definition 3.5. A sequence (x_n) is *bounded* if there exists a number $M>0$ such that $\mid x_n \mid \leq M$ for all $n \in \mathbf{N}$. Geometrically, this means we can find an interval $[-M,M]$ that contains every term in the sequence (x_n).

Example 3.6. Given the sequence $x_n = (1,\ 2,\ 1,\ 2,\ 1,\ 2\ ...)$, we can see that the interval $[1, 2]$ contains every term in x_n. This sequence is therefore a *bounded* sequence.

Example 3.7. Given the sequence $x_n = (10,\ 100,\ 1000,\ 10000,\ ...)$, we can see that there is no real number that serves as an upper bound because $\lim(x_n)$ is infinity. Therefore, there does not exist any interval that contains every term in the sequence x_n, and x_n is *not* a bounded sequence.

Theorem 3.8. *Every convergent sequence is bounded.*

Example 3.9. Theorem being illustrated:
Let $x_n = \frac{n+1}{n}$, which is the following sequence:

$$\left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4} ...\right) \tag{3.10}$$

We know this converges to 1 and can verify this using the same logic used in the proof under the definition of convergence showing that $\frac{1}{n}$ converges to zero. Therefore, as n becomes very large, x_n approaches 1, but is never equal to 1. By the above theorem, we know that this sequence is bounded because it is convergent. We can see that x_n is a decreasing sequence, so the x_1 is the largest value of the sequence and is the "upper bound." The limit of the sequence, 1, is the lower bound. An interval that contains every term in the sequence x_n is (1,2].

4. Continuity

Theorem 4.1. *If f: $\boldsymbol{R} \to \boldsymbol{R}$ is continuous, $x_n \to x$ implies f(x_n) $\to$ f(x)*

Example 4.2. Theorem being applied:
Let f(x) = $3x$. This function is continuous. Let $\lim(x_n) = 5$. In order words, $x_n \to 5$. By the above theorem, this implies that f(x_n) $\to$ f(5). This is equal to $3x_n \to (3)(5)$ which is also equal to $3x_n \to 15$. Therefore, we are able to see what the limit of f(x_n) is using this theorem.

Example 4.3. Theorem failing when function is non-continuous:
Let f(x) be $\frac{1}{x}$, a non-continuous function. We know this is non-continuous because there is an asymptote at x=0. Let x_n be $\frac{1}{n}$. We know this converges to zero based on a previous proof. Let's see if the continuity theorem fails for a non-continuous function f. The theorem states that f(x_n) converges to f(x) if $x_n \to x$. We know that $x_n \to 0$, so if the theorem works, then f(x_n) $\to$ f(0). But f(0) = $\frac{1}{0}$ which does not exist. Therefore, f(x_n) cannot converge to 1/0, and the theorem fails for this non-continuous function.

5. Subsequences and the Bolzano-Weierstrass Theorem

Definition 5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 ...$ be an increasing sequence of natural numbers. Then, the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4} ...)$ is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbf{N}$ indexes the subsequence.

Example 5.2. Let $x_n = \frac{1}{n} = (1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\ ...)$. Below are two examples of valid subsequences:

$$(\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}...) \tag{5.3}$$

$$(\frac{1}{20}, \frac{1}{200}, \frac{1}{2000}...) \tag{5.4}$$

Theorem 5.5. *Bolzano-Weierstrass Theorem:*
Every bounded sequence contains a convergent subsequence.

Example 5.6. Given a sequence x_n = (1,2,3,4,1,2,3,4...), a convergent subsequence can be found.

Proof. We know that this sequence is bounded by the interval $[1, 4]$. By the Bolzano-Weierstrass Theorem, we can say that there indeed exists a convergent subsequence of x_n. Just by looking at this sequence, we can see four convergent subsequences: (1,1,1...), (2,2,2...), (3,3,3...), and (4,4,4...). These subsequences converge to 1, 2, 3, and 4 respectively. □

Example 5.7. Given an unbounded sequence x_n = (1,2,3,4,5...), a convergent subsequence of x_n does not exist

Proof. A convergent subsequence does not necessarily exist because this sequence does not satisfy the Bolzano-Weierstrass Theorem. Recall that any subsequence of a sequence is non-repeating and in the order of the original entries of x_n. Notice that x_n is increasing for all values of n and is divergent, considering the sequence continues until infinity. Therefore, for any subsequence a_n, the values will be increasing toward infinity as well, and the subsequence will also be divergent. □

Theorem 5.8. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Example 5.9. Let us return to the example of a divergent sequence that was given under the definition of divergence. Recall that this sequence, x_n, was (1, -1, 1, -1, 1, -1...). One subsequence of x_n is (1, 1, 1, ...). This subsequence converges to 1. Another subsequence of x_n is (-1, -1, -1, ...). This subsequence converges to -1. Now, we will prove that x_n is divergent by contradiction. Assume x_n is convergent. Then, by the above theorem, all its subsequences converge to $\lim(x_n)$, implying that all its subsequences converge to the same value. The two subsequences of x_n stated above converge to different values. Therefore, this contradicts our original hypothesis that x_n is convergent. We are then able to conclude that x_n is divergent.

Acknowledgements. I would like to thank my two mentors, Sean Howe and Abby Ward for meeting with me each week for the duration of my stay at the program. Thank you so much for suggesting this topic, explaining everything to me that I was confused about, editing and helping with my paper, and calming me down when I got overwhelmed. I would also like to thank Peter May for organizing this wonderful REU and for giving me the opportunity to participate.

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