BONNET'S THEOREM AND VARIATIONS OF ARC LENGTH

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Abstract. This paper aims to give a basis for an introduction to variations of arc length and Bonnet's Theorem. To do this, it will begin by defining the first and second fundamental forms of surfaces and by using them to define Gaussian curvature. With those terms defined, we will then briefly explore the Theorema Egregium, Geodesics, and the Gauss-Bonnet theorem. Subsequently, the exponential map and completeness will be explored in the interest of proving Hopf-Rinow. Finally, with that background finished, this paper will provide an introduction to Variations of Arc length. With that as a basis, we will prove Bonnet's theorem and briefly explore some mathematical consequences of the theorem.

Contents

1. Introduction

Differential Geometry on Smooth Surfaces is a fascinating mathematical discipline that uses techniques from calculus and linear algebra to analyze surfaces where derivatives can be defined. This paper will endeavor to provide a brief introduction to differential geometry on smooth surfaces specifically focusing on exploring the first and second variations of Arc Length and Bonnet's theorem. To this end, we will first provide a rigorous definition of regular surfaces and explore the first and second fundamental forms. These concepts give us the foundational definition of Gaussian Curvature, a scalar that expresses how a surface is curled or flares out. That will be the basis used to explore the Theorema Egregium, an introduction to Geodesics, and the famous Gauss-Bonnet theorem, which states:

Date: August 17.

Theorem 1.1 (Gauss-Bonnet Theorem; see [2, Thm. 4.7])**.** *The integral Gaussian curvature over a closed surface is a multiple of 2π.*

A previous REU paper by Chase [2] covers Gauss-Bonnet in greater detail. These results provide a basis for a basic study of differential geometry on smooth surfaces. We can build an understanding of variations of arc length, in which we study the arc lengths of families of curves, and use that to prove Bonnet's theorem, an important result in differential geometry that states the following:

Theorem 1.2 (Bonnet's Theorem, see Theorem 8.1 below)**.** *If the Gaussian curvature K of a surface is bounded below by some $\delta > 0$, then S is compact and has a diameter of at most $\frac{\pi}{\delta}$.*

To prove this, we will build a definition of completeness using the exponential map and subsequently use that to prove Hopf-Rinow, the theorem that there always exists a minimal geodesic between two points on a regular surface. This paper will then proceed to define and elucidate the first and second Variations of arc length, those being facts about families of curves. Finally, this paper will conclude by proving Bonnet's theorem and then briefly exploring some mathematical consequences of it.

2. Surfaces, Fundamental Forms, and Gaussian Curvature

This paper will first begin by delineating the mathematical objects that are to be examined. This section will detail the mathematical objects that this paper is concerned with, Regular Surfaces, and then explore the first fundamental form: the inner product on the tangent space of a regular surface.

Definition 2.1. A set $S \subset R^3$ is a Regular surface if, for each $p \in S$, there exists a neighborhood $V \subset R^3$ and an open set $U \subset R^2$ such that there exists a map $f : U \to V \cap S$ that satisfies the following properties:

(1) f is infinitely differentiable.
(2) f is a homeomorphism.
(3) (The Regularity Condition) For each $q \in U$, the differential df_q is one to one.

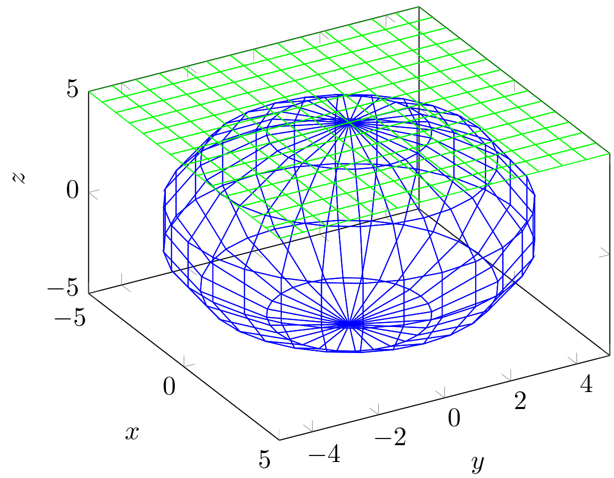
One will note that (1) is a requirement for a surface to have if one wishes to do differential geometry on it due to differential geometry being based on derivatives. We add (2) in order to make S homeomorphic with R^2, making S a 2-dimensional manifold, and we add (3) to make S diffeomorphic with R^2. Additionally, (3) guarantees a tangent plane at every point on the regular surface — observe that if one puts f in the form $f(u, v) = (x(u, v), y(u, v), z(u, v))$ then the regularity condition is stating that the Jacobian matrices, $\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(x,z)}{\partial(u,v)}$, are not all zero at point p.

We will now delve into some examples to elucidate what a regular surface is and isn't.

Example 2.2.

(1) An example of a regular surface is a sphere. One will note that there exists different parameterizations, such as $z = \sqrt{1 - x^2 - y^2}$ among others, that

partially cover the sphere and that are infinitely differentiable, homeomorphisms, and satisfy the regularity condition. This can be shown visually by the fact that the sphere is differentiable all around and that at every point on the sphere there exists a single tangent plane.



This is a graph of a sphere that shows the tangent plane at $(0, 0, 5)$.

(2) An example of a surface that fails condition 2 would be the surface defined by the function $f(u, v) = (u, v, z(u, v))$ where $z = 1$ when either u or v is rational and $z = 0$ when u and v are irrational. f is clearly not continuous, so the surface is not a regular surface.
(3) An example of a surface which fails condition 3 would be a cone. The Jacobian matrices are all zero at the vertex; the definition of a derivative when applied at the vertex of the cone does not give a 2 dimensional vector space like it would when the Jacobian matrices are not all zero.

We shall now define the first fundamental form.

Definition 2.3. Let $f(u, v) : U \subset R^2 \to S$, where S is a regular surface, be a map. The first fundamental form of a surface is the expression $I = r \cdot r$ where $r = f_u dv + f_v du$.

The first fundamental form is the Euclidian dot product of this velocity vector with itself. Because this is an intrinsic property of a surface, the First Fundamental form does not depend upon the parametrization of the surface. Observe that if one multiplies out $r \cdot r$, then one gets the equation
$r \cdot r = (f_u \cdot f_u)(du)^2 + 2(f_u \cdot f_v)(dudv) + (f_v \cdot f_v)(dv)^2$.
The various inner products are often denoted $E = f_u \cdot f_u$, $F = f_u \cdot f_v$ and $G = f_v \cdot f_v$. From this equation, we can conclude that the first fundamental form is a quadratic form.

Here is an example of the first fundamental form being calculated.

Example 2.4. Consider the surface parametrized by the equation $f(u, v) = \{u^2, v^2, uv\}$. The first fundamental form of this surface would be:

$I_{f(u,v)} = (4u^2 + v^2)(du)^2 + 2(uv)dudv + (4v^2 + u^2)(dv)^2$.

An important attribute of the first fundamental form is its connection to arc length. To explain this connection, arc length will first be defined.

Definition 2.5. Let $f : I \to R^3$ be a regular parametrized curve. The arc length of the curve f for $t \in I$ would be:
$s(t) = \int_{t_0}^{t} \mid f(t) \mid dt$
where $f'(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$.

With that definition, we can algebraically show the connection between the first fundamental form and arc length.

$$\begin{aligned}\int_a^b \mid f'(t) \mid dt &= \int_a^b \sqrt{f'(t)f'(t)}dt \\ &= \int_a^b \sqrt{(f'(t)_u du + f'(t)_v dv)(f'(t)_u du + f'(t)_v)}dv \\ &= \int_a^b E(u'(t))^2 + F(u'(t)v'(t)) + G(v'(t))^2 dt\end{aligned}$$

Thus, one can calculate the arc length of a curve using the first fundamental form of a surface. This relation between the first fundamental form and arc length also naturally provides information about surfaces that are isometric to each other. An isometry is a homeomorphic map of curves from one surface to another that respects arc length. The following theorem is the relation between isometries and the first fundamental forms of surfaces.

Theorem 2.6. *If $S \subset R^3$ and $T \subset R^3$ are two regular surfaces, then coordinate patches of S and T are isometric if and only if there exist parameterizations $f : U \subset S \to R^3$ and $g : U' : S' \to R^3$ such that their first fundamental forms are equal.*

An important theorem relating the first fundamental form of surfaces with isometries and the definition of area will now be presented.

Definition 2.7 (see [2, Thm.1.5])**.** If $R \subset S$ is a bounded region of regular surface S and is contained in the parametrization $f : U \subset R^2 \to S$ then the positive number
$\iint_Q |f_u \times f_v| du\, dv$ where $Q = f^{-1}(R)$
is the area of R.

There are numerous geometric and algebraic justifications of this definition that will not be presented in this paper in the interest of space. Some justifications can be seen in Chase and Do Carmo.

3. The Second Fundamental Form and Gaussian Curvature

We will now explore the Second Fundamental Form and Gaussian Curvature, including Gauss's Theorema Egregium. All definitions in this section are paraphrasings from Chapter 3 of do Carmo, so that chapter can be consulted for more information.

Definition 3.1 (The Gauss Map)**.** Consider a parametrization $f : U \subset R^2 \to S$ with regular surface S at point $p \in S$. There exists a map $N : f(U) \to S^2$, with S^2 being the unit sphere, such that
$N(q) = \frac{f_u \times f_v}{|f_u \times f_v|}(q)$ for any $q \in f(U)$

The Gauss Map is essentially a map of all the unit normal vectors of a surface. Note that the fact that $N(q)$ exists at any point in the image is a non-trivial statement. It will not be proven in this paper, but a proof is provided in do Carmo. When one has an orientable surface S, where N can be defined for all points in S, this map N is called the Gauss map of a surface. This map is the map of unit normal vectors of every point in the image of f. Essentially, each unit normal vector on that curve is transported to the origin, and that vector ends at a point on the unit sphere.

From this point on, any regular surface S will also be assumed to be orientable. A few facts about the Gauss Map will now be presented without proof. Proofs for these statements can be found in do Carmo, 3-2 [1].

- The Gauss Map is differentiable and the differential dN_p at any point $p \in S$ is a linear map.
- The differential map dN_p is a self adjoint linear map, meaning that $dN_p(v) \cdot w = v \cdot dN_p(w)$ for any $v, w \in T_p(S)$.

Now the Second Fundamental Form will be defined:

Definition 3.2. The Second Fundamental Form II_p is defined in $T_p(S)$ by $II_p(v) = -(dN_p(v) \cdot v)$.
The Second Fundamental Form is a quadratic form, and can be written as:
$II_p = O(du)^2 + Pdudv + Q(dv)^2$
where $O = r_{uu} \cdot N$, $P = r_{uv} \cdot N$, and $Q = r_{vv} \cdot N$.

Note that because dN_p is self adjoint, II_p is a quadratic form. A further explanation of this can be found in do Carmo. Note that the second fundamental form informally is how quickly a surface pulls away from a tangent plane at a point.

Before we can understand Gaussian Curvature, we must first understand the normal curvature. The normal curvature is as follows:

Definition 3.3. Let $C \subset S$ be a regular curve such that $p \in C$. If k is the curvature of C at p and $cos\theta = n \cdot N$ where N is the normal vector to C at p and N the normal vector to S at p, the number $k_n = kcos\theta$ is the normal curvature of C at p.

Observe that , if C is a curve parametrized by function α such that s is the arc length of C and $\alpha(0) = p$, then

$$\begin{aligned} II_p(\alpha) &= -(dN_p(a'(0)) \cdot \alpha(0)) \\ &= (N'(0) \cdot \alpha''(0)) \\ &= N \cdot kn(p) \\ &= k_n(p) \end{aligned}$$

Thus, the second fundamental form evaluated on a vector v in $T_p(S)$ is the normal curvature of a regular curve passing through p and tangent to the velocity vector v.

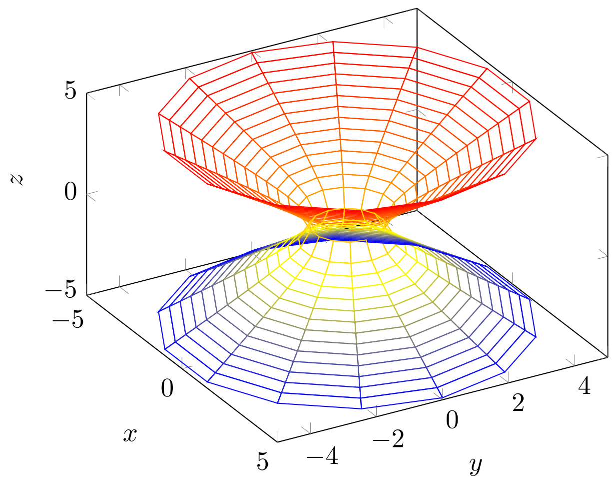
We will now explain Gaussian Curvature.

Definition 3.4. Let dN_p be the differential of the Gauss map. The determinant of dN_p is the Gaussian curvature K of S at $p \in S$.

A few examples will be given to better understand Gaussian Curvature:

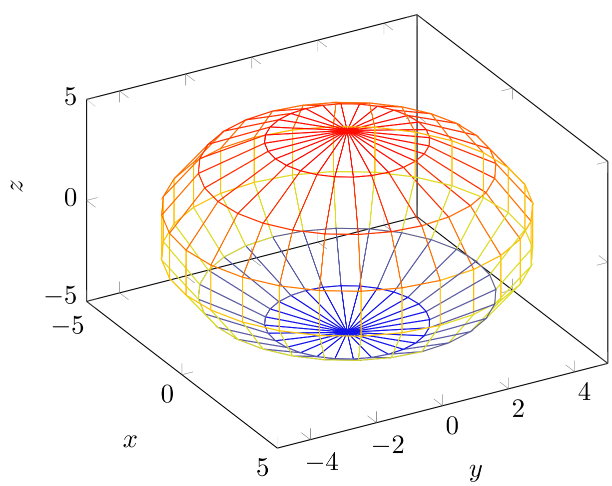
Example 3.5.

- A hyperboloid is an example of a surface that has a negative gaussian curvature. Surfaces that flare out like hyperboloids have negative gaussian curvature.



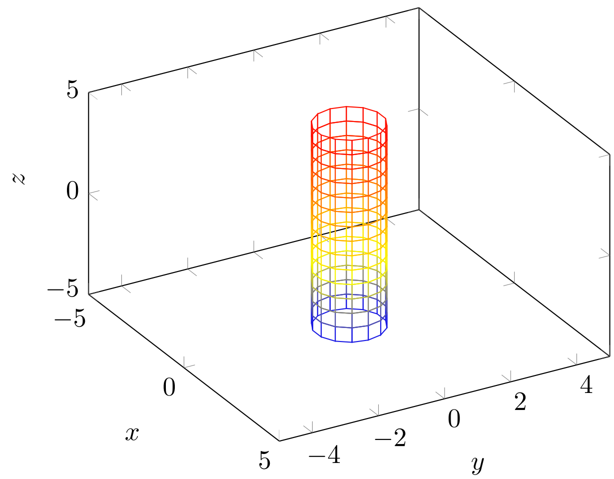
A hyperboloid has negative Gaussian curvature.

- A sphere is an example of a surface with positive Gaussian Curvature. One would get that $n = \frac{1}{a}r$ where r is the radius of the sphere. One will observe that $O = \frac{1}{r}E$, $P = \frac{1}{r}F$, and $Q = \frac{1}{r}G$. Thus, the gaussian curvature of of a sphere would be $\frac{1}{r^2}$, this being positive Gaussian Curvature. Surfaces that curve in like a sphere have positive Gaussian Curvature.



A sphere has positive Gaussian curvature.

- A cylinder is an example of a surface with 0 gaussian curvature. This is clear when one notices that one can twist a cylinder into a plane without stretching it.



A cylinder has 0 Gaussian Curvature.

We will now go into one of the most important theorems in Differential Geometry: Gauss's Theorema Egregium.

Theorem 3.6 (Theorema Egregium)**.** *Gaussian Curvature depends only on the first fundamental form.*

Proof. While the proof is too long to state in full due to space constraints, a general outline of the proof will be given. It can be shown that, given a smooth surface and an orthonormal basis $\{e^1, e^2\}$, that

$K = \frac{e^1_u \cdot e^2_v - e^1_v \cdot e^2_u}{\sqrt{EG-F^2}}$

and that

$e^1_u \cdot e^2_v - e^1_v \cdot e^2_u$

is solely dependent upon the first fundamental form. Thus, K is solely dependent upon the first fundamental form. □

Note that more information on this proof can be found in both do Carmo and in Chase's paper.

This statement classifies the Gaussian Curvature of a surface as an invariant; it does not depend upon what parametrization on chooses. It states that, without distorting or stretching, one cannot transform one shape of Gaussian curvature K_1 to a shape of Gaussian curvature K_2. A well known example of this phenomenon would be the fact that because the roughly spherical Earth and a planar map have different Gaussian curvatures, any map must have deformations.

4. Geodesic Curves

Now it is time to examine another concept in Differential Geometry that is needed to prove Bonnet's theorem: Geodesic Curves. A good way to intuitively understand geodesic curves is that they are the shortest path between points on a surface. To begin our brief study of them, we must first define the covariant derivative:

Definition 4.1. Let w be a differentiable vector field in an open set $U \subset S$ and $p \in U$. Let α be a parametrized curve such that $\alpha(0) = p$ and $\alpha'(0) = y$ and let $w(t)$ be the restriction of w to α. The projection of $w'(0)$ onto the plane $T_p(S)$ is denoted by $\frac{Dw}{dt}(0)$.

It is important to note that the covariant derivative is in the intrinsic geometry of a surface. This can be proven by showing that $\frac{Dw}{dt}$ depends only on the vector y and is dependent on the first fundamental form.

Another concept that will be needed to understand geodesics is regularity.

Definition 4.2. If a curve $f : L = [0, l] \to S$ is the restriction to L of a differentiable mapping of $L \subset R$ to S, then if $f(0) = p$ and $f(l) = q$, then f joins p to q and f is regular if $f'(t) \neq 0$ for any $t \in [0, l]$.

Simply put, a regular curve is a curve that joins two points and always has a non-zero derivative. With these two definitions, we can now explore a definition for geodesics.

Definition 4.3. A nonconstant, parametrized curve $\gamma : I \to S$ is geodesic at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t. This means the covariant derivative, $\frac{D\gamma'(t)}{dt} = 0$. We say γ is a parametrized geodesic if it is geodesic for all $t \in I$.

Another, equivalent, definition of a geodesic is as follows:

Definition 4.4. A regular connected curve $C \in S$ is a geodesic if, for every $p \in S$, there eists a parametrization $f(s)$ such that, with s being the arc length parameter, $f'(s)$ is a parallel vector field along $f(s)$.

Both of those definitions are equivalent and can be used at different times.

The following proposition that will be used in Gauss Bonnet will be presented without proof. Proof of this proposition can be found in do Carmo.

Proposition 4.5. *Let $x(u, v)$ be an orthogonal parametrization of a neighborhood of an oriented surface S and let $w(t)$ be a differentiable field of unit vectors along the curve $x(u(t), v(t))$. It follows that,*

$\frac{Dw}{dt} = \frac{1}{2\sqrt{EG}} G_u \frac{dv}{dt} - E_v \frac{du}{dt} + \frac{d\beta(t)}{dt}$

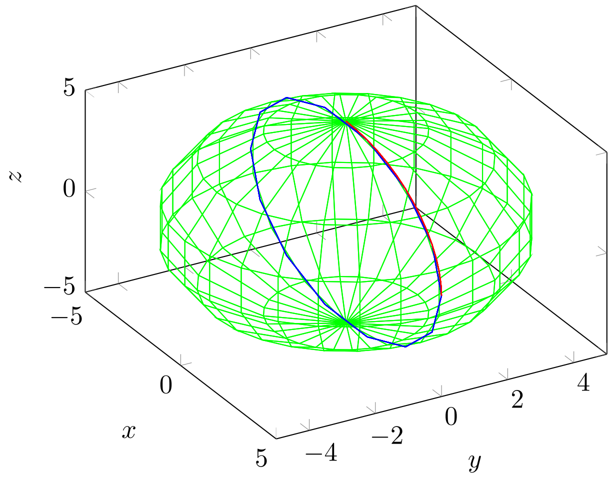
Where $\beta(t)$ is the angle from x_u to $w(t)$ in a given orientation.

Additionally, another concept that must be introduced is that of geodesic curvature.

Definition 4.6. Let k_g be the geodesic curvature of a smooth curve on a surface.

$k_g = t' \cdot (n \cdot t)$.

The geodesic curvature of a smooth curve essentially measures how far from a geodesic a curve is. For example, the curvature of geodesics, such as a great circle on a sphere, is zero.



The geodesic between the points $(0, 0, 5)$ and $(5, 0, 0)$ would be the red curve. That red curve can be seen to be an arc of the blue great circle. Because this curve is a geodesic, its geodesic curvature would be 0.

Definitions 4.1-4.4 and proposition 4.5 were based on do Carmo, section 4-4, so more information on Geodesics can be found there.

5. Gauss-Bonnet and Immediate Applications

It is now time for one of the most important theorems in all Differential Geometry: Gauss-Bonnet. There are two different versions of the theorem that will be dealt with in this paper: the local version and global version. The local version is sufficient to provide the results needed for Bonnet's Theorem, so the global theorem will be presented without proof.

We will first establish a few definitions before we get into the theorem.

Definition 5.1. Let $\alpha[0, l] \to S$ be a curve such that S is a regular surface. α is a simple, closed piecewise, regular, parametrized curve if:

(1) $\alpha(0) = \alpha(l)$.
(2) $t_1 \neq t_2, t_1, t_2 \in [0, l]$ implies $a(t_1) \neq a_(t_2)$.
(3) There exists a subdivision
$0 = t_0 < t_1 < ... < t_k = l$
where α is differentiable and regular on any interval $[t_i, t_{i+1}]$ for any $i = 0, ..., k - 1$. The trace of α on these integrals are called the arcs of α.

The above definition essentially states that α is a closed curve without self intersections and that has a well defined tangent line at all but a finite number of points. Additionally, the point $\alpha(t_i)$ are called the vertices of α. This definition will prove useful in proving Gauss-Bonnet.

Definition 5.2. A region $R \subset S$ is a simple region of S if R is homeomorphic to a disk and the boundary of ∂R of R is the trace of a simple, closed, piecewise regular, parametrized curve $\alpha : I \to S$.

The curve α is positively oriented if the positive orthogonal basis $\alpha'(t), h(t)$ satisfies that $h(t)$ points towards R.

By definition, R is a simple region if it is homeomorphic to a disc and if it is bounded by a simple, closed, piecewise regular, parametrized curve.

Definition 5.3. The integral of f over the region R is the integral
$\iint_{x^{-1}R} f(u,v)\sqrt{EG-F^2}dudv = \iint_R fd\sigma$

Note that the integral is not dependent upon the parametrization x.

With those definitions, we can now state the local version of Gauss-Bonnet. In the interest of space, the theorem will not be given a full proof, but the proof will get an outline.

Theorem 5.4 (Gauss-Bonnet (Local))**.** *Let $x : U \to S$ be an orthogonal parametrization of an oriented surface S, where $U \subset R^2$ is homeomorphic to an open disc. Let $R \subset x(U)$ be a simple region of S and let $\partial R = \alpha(I)$ for some positively oriented curve α that is parametrized by arc length s. Let $\alpha(t_0), ..., \alpha(t_k)$ and $\theta_0, ...\theta_k$ be the vertices and external angles of α. Then*
$\sum\limits_{i=0}^{k} \int_{t_i}^{t_{i+1}} k_g(s)ds + \iint_R Kd\sigma + \sum_{i=0}^{k}\theta_i = 2\pi.$
with $k_g(s)$ being the geodesic curvature of the regular arcs of α and K is the gaussian curvature of S.

Proof. Due to being outside the focus of this paper, variations of arc length and Bonnet's theorem, this proof will not be given in full. An outline of the proof will be given, though. This proof begins by using proposition 4.5 to state,
$\frac{Dw}{dt} = \frac{1}{2\sqrt{EG}}G_u\frac{dv}{dt} - E_v\frac{du}{dt} + \frac{d\beta(t)}{dt}$

Note how proposition 4.5 is what allows us to link the first fundamental form to the external angles of α. If that expression is integrated in every interval $[t_i, t_{i+1}]$ for $i = 0, ..., k-1$ and those integrals added, then we get the equation,
$\sum\limits_{i=0}^{k} \int_{t_i}^{t_{i+1}} k_g(s)ds = \sum\limits_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{G_u}{2\sqrt{EG}}\frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}}\frac{du}{ds}ds + \sum\limits_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\kappa_i}{ds}ds.$

Where κ_i are a group of differentiable functions that measure in each interval the positive angle from x_u to $\alpha'(t)$. With that equation, the Gauss-Green theorem can be applied to the equation. The Gauss-Green theorem combined with the Gauss formula for $F = 0$ allows us to state that
$\iint_{x^{-1}(R)} (\frac{E_v}{2\sqrt{EG}})_v + (\frac{G_u}{2\sqrt{EG}})ududv = -\iint_R Kd\sigma$

By using the theorem of turning tangents, a theorem that states that the cumulation of the angle of the tangent vector to α with any given direction is equal to 2π, we can state that
$\sum\limits_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\kappa_i}{ds}ds = \pm 2\pi - \sum\limits_{i=0}^{k} \theta_i.$

Because α is positively oriented, the sign for 2π would be a plus. With that, we can put all the information together and get the equation,
$\sum\limits_{i=0}^{k} \int_{t_i}^{t_{i+1}} k_g(s)ds + \iint_R Kd\sigma + \sum_{i=0}^{k}\theta_i = 2\pi$ □

This theorem makes the Gaussian curvature to be a topological invariant of a region within a surface. A corollary of Gauss-Bonnet will now be presented.

Corollary 5.5. *If R is a simple region of S then*
$\sum\limits_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds + \iint_R Kd\sigma + \sum\limits_{i=0}^{k}\theta_i = 2\pi$

We will now present the global version of Gauss-Bonnet without proof. More information can be found in both do Carmo and Chase.

Theorem 5.6 (Gauss-Bonnet (Global)). *Let $R \subset S$ be a regular region of an oriented surface and let $C_1, ..., C_n$ be the closed, simple, piecewise, regular curves which form the boundary ∂R. Suppose that each C_i is positively oriented and let $\theta_1, ..., \theta_p$ be the set of all external angles of the curves $C_1, ..., C_n$. It follows that*
$\sum\limits_{i=1}^{n} \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum\limits_{l=1}^{p} \theta_i = 2\pi\chi(R).$
where s is the arc length of C_i and $\chi(R)$ is the Euler characteristic of R.

All definitions and theorems in this section were based off of do Carmo, section 4-5. More information on Gauss-Bonnet can be found in both do Carmo and in Chase.

6. Completeness and Hopf-Rinow

This section will be dedicated towards the introduction of the concepts of completeness and the proving of the Hopf-Rinow theorem: that for any two points on a complete surface, there exists a minimal geodesic connecting them. To do this, we will first introduce the concept of the exponential map, use that to help define what a complete surface is, present a distance function on smooth surfaces, present a few facts about distance, and then proove Hopf-Rinow.

Definition 6.1. Let $p \in S$ and let $v \in T_p(S)$ such that v is a nonzero vector. The exponential map is the unique geodesic $\exp_p : (-\epsilon, \epsilon) \to S$ with $\exp_p(0) = p$ and $\exp_p'(0) = v$. $\exp_p(v) = \gamma(l, v)$ and $\exp(0) = p$.

Note that the existence of this map is a consequence of Gauss-Bonnet.

While we have achieved many important results so far, we need to restrict the type of surface analyzed in order to prove global theorems such as Hopf-Rinow and Bonnet's theorem. To do this, we will introduce the concept of complete surfaces using a definition from do Carno.

Definition 6.2. A regular surface S is said to be complete when for every point $p \in S$, any parametrized geodesic $\gamma : [0, \epsilon) \to S$ of S, starting from $p = \gamma(0)$, may be extened into a parametrized geodesic $\overline{\gamma} : R \to S$ defined on the entire line R.

Equivalently, S is complete when for every $p \in S$, the mapping exp_p is defined for every $v \in T_p(S)$.

Here are some examples of complete surfaces:

Example 6.3. '

- A sphere is an example of a complete surface. The great circles that are its geodesics can be extened to include the whole number line.
- A plane is an example of a complete surface. The lines that are its geodesics can be extended to include the whole number line.
- A cone without its vertex is not a complete surface becaues any geodesic that is fully extended will have to reach the vertex.

Another important concept is the idea of intrinsic distance.

Definition 6.4. The distance, for $p, q \in S$, $d(p, q) = \inf\{\alpha_{p,q}\}$ where α is a piecewise differentiable curve joining p to q.

A few interesting properties of complete surfaces will now be presented without proof. Proofs for these propositions can be found in 5-3 of do Carmo. Note that S is a regular, connected surface.

Proposition 6.5. *Given two points $p, q \in S$, there exists a parametrized piecewise differentiable curve joining p to q.*

Proposition 6.6. *The distance function has the following properties:*

(1) $d(p,q) = d(q,p)$
(2) $d(p,q) + d(q,r) \geq d(p,r)$
(3) $d(p,q) \geq 0$
(4) $d(p,q) = 0 \Leftrightarrow p = q$

Proposition 6.7. *A topologically closed surface $S \subset R^3$ is complete.*

Corollary 6.8. *A compact surface $S \subset R^3$ is complete.*

Theorem 6.9 (Hopf -Rinow)**.** *Let S be a complete surface. Given two points, $p, q \in S$, there exists a minimal geodesic joining p to q.*

Proof. Let $B(0) \in T_p(S)$ be a disc of radius β, centered in the origin of the tangent plane and contained in a neighborhood $U \subset T_p(S)$, where $\exp_p$ is a diffeomorphism. Note that the boundary of $\exp_p(B(0))$ is compact because it is the continuous image of the compact set $B(0)$.

Consider the geodesic $\gamma(s) = exp_p(sv)$ such that s is arc length. If $\gamma(r) = q$, then γ is a geodesic. This can be done on a case by case basis. $\square$

Thus, we have established that for any two points on a complete surface there exists a geodesic joining. That will be essential to proving Bonnet's Theorem.

7. Varations of Arc Length

The final concepts needed to prove Bonnet's Theorem come from Variations of Curves. We will first define what a variation is: a map of sister curves that go alongside an original curve. Subsequently, we will define an arc length dependent function L in order to compare the arc length of various curves. We will then conclude this section by proving some mathematical facts about L, the derivatives of L, and its relation to Geodesics. Specifically, the relationship between L' and geodesics (Prop 7.9) and the relationship between L'' and Gaussian Curvature (Prop. 7.12). These propositions and lemmas will be the final foundation needed to prove Bonnet's theorem.

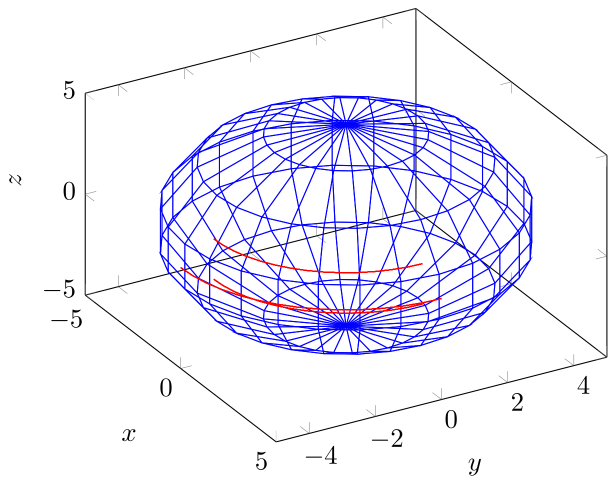
Definition 7.1. Let $\alpha : [0, l] \to S$ be a regular parametrized curve where $s \in [0, l]$ is the arc length. A variation of α is a differentiable map such that

$h(s, 0) = \alpha(s)$ for $s \in [0, 1]$.

A variation $h_t = h(s, t)$ is proper if

$h(0, t) = \alpha(0)$ and $h(l, t) = \alpha(l)$ for any $t \in (-\epsilon, \epsilon)$.

Essentially, a variation of α is a group of curves that goes side by side with α. The variation is proper if all the curves begin and end at the same point. To help visualize the concept, a graphical representation of a variation will be given.



This picture shows three curves in a variation of the curve $f(u) = (5cos(u), 5sin(u), 0)$. Note that this is a variation if $h(u, 0) = f(u)$ and that this variation is improper because for the two other curves on this graph, their endpoints do not match $f(0)$ and $f(l)$.

Another definition will be given.

Definition 7.2. Let h be a variation α. the differentiable vector field V is defined as

$V(s) = \frac{\partial h}{\partial s}(s, 0)$.

An interesting proposition immediately follows from these two definitions.

Proposition 7.3. *Let $V(s)$ be a differentiable vector field along a parametrized regular curve $\alpha : [0, l] \to S$. It follows that there exists a variation $h : [0, l] \times (-\epsilon, \epsilon) \to S$ such that $V(s)$ is the variational vector field of h and if $V(0) = V(1) = 0$, then h can be chosen to be proper.*

Proof. This proof is done by first showing that there exists some $\delta > 0$ such that if $\mid v \mid < \delta$, then $v \in T_{a(s)}(S)$, then $\exp_{a(s)} v$ is well defined for $s \in [0, l]$. Subsequently, a geodesic can be created such that $V(0) = V(l) = 0$. This shows that h is proper. □

In order to compare the arc length of the original curve, which is equal to h_0, with other variations, we will define a function $L : (-e, e) \to R$ such that:

$L(t) = \int_0^t \mid \frac{\partial h}{\partial s}(s, t) \mid ds$.

The study of L in a neighborhood of $t = 0$ will tell us about the arc length and how it behaves in curves near α. A few lemmas needed to prove facts about L will now be presented.

Lemma 7.4. *The function L is differentiable in a neighborhood of $t = 0$ and the derivative can be found by differentiation under the integral sign.*

Proof. Because h is parametrized by arc length, $\mid \frac{\partial h}{\partial s} \mid = 1$. Because $[0, l]$ is compact, it follows there exists a $\delta > 0$ such that

$\frac{\partial h}{\partial s}$ is nonzero for $s \in [0, 1]$ and $\mid t \mid < \delta$.

Because the absolute value of a nonzero differentiable function is differentiable, L is differentiable for $\mid t \mid< \delta$. Additionally, there exists a calculus theorem such that,

$L'(t) = \int_0^t \frac{\partial}{\partial t} \mid \frac{\partial h}{\partial s} \mid ds.$ □

Two more Lemmas will be presented that describe the behavior of vector fields.

Lemma 7.5. *Let $w(t)$ be a differentiable vector field along the parametrized curve α and let $f : [a, b] \to R$ be a differentiable function. It follows that*
$\frac{D}{dt}(f(t)w(t)) = f(t)\frac{Dw}{dt} + \frac{df}{dt}w(t).$

Proof. One can simply note that the covariant derivative is the tangential component of the usual derivative. It is then simple calculus from there to show the lemma. □

The following lemma has a similar proof to the previous one.

Lemma 7.6. *Let $v(t)$ and $w(t)$ be differentiable vector fields along α. It follows that*
$\frac{d}{dt}(v(t) \cdot w(t)) = (\frac{Dv}{dt} \cdot w(t)) + (v(t) \cdot \frac{Dw}{dt}).$

The next Lemma will be presented without proof in the interests of time, but a proof can be found in do Carmo, section 5-4.

Lemma 7.7. *Let $H : [0, l] \times (-\epsilon, \epsilon) \to S$ be a differentiable map. It follows that*
$\frac{D}{\partial s}\frac{\partial h}{\partial t} = \frac{D}{\partial t}\frac{\partial h}{\partial s}.$

We can now compute the first derivative of L at $t = 0$.

Proposition 7.8. *Let h be a proper variation of the curve α and let $V(s) = \frac{\partial h}{\partial t}$ be the variational vector field of h. Then:*
$L'(0) = -\int_0^t A(s) \cdot V(s)ds$ *where* $A(s) = (\frac{D}{\partial s})(\frac{\partial h}{\partial s}).$

Proof. If $t \in (-\delta, \delta)$, then Lemma 1 gives us:
$L'(t) = \int_0^t \frac{\partial}{\partial t} \mid \frac{\partial h}{\partial s} \mid ds.$
Using Lemmas 3 and 4 we get:
$L'(t) = \int_0^t \frac{(\frac{D}{\partial s}\frac{\partial h}{\partial t} \times \frac{\partial h}{\partial s})}{\frac{\partial h}{\partial s}} ds.$
Because $\mid \frac{\partial h}{\partial s} \mid= 1$ at $(s, 0)$, it follows that
$L'(0) = \int_0^t \frac{D}{\partial s}\frac{\partial h}{\partial t} \times \frac{\partial h}{\partial s} ds.$
By applying Lemma 3 once more, we get
$L'(0) = -\int_0^t \frac{D}{\partial s}\frac{\partial h}{\partial s} \cdot \frac{\partial h}{\partial s} ds.$ □

Note that A is called the acceleration vector of a curve α and its norm is the absolute value of the geodesic curvature of α. Additionally, if h is improper, then the formula would be the second equation.

An interesting consequence of this Proposition is the following:

Proposition 7.9. *A regular parametrized curve α where the parameter s is the arc length of α, is a geodesic if and only if for every proper variation h of α, $L'(0) = 0$.*

Proof. Because the acceleration, A, of a geodesic is zero, the only if part is automatically true.

If we have $L'(0) = 0$ for every proper variation of α, then we can create a variation corresponding to the vector field $V(s) = f(s)A(s)$. It can then be shown algebraically that $A(s) = 0$. It follows that $L'(0) = 0$. Thus, the proposition is satisfied. □

We now have a mutual dependence relation between geodesic curves and L. From now on, we will only talk about proper variations of geodesics γ parametrized by arc length. Additionally, we will now refer to orthogonal variations where the variational field V satisfies $V(s) \cdot \gamma'(s) = 0$.

We will now find how to compute $L''(0)$. To do this, we will present two lemmas and then a proposition. The two lemmas will be presented without proof due to space constraints, but proofs can be found in do Carmo. These statements will provide the final groundwork needed to prove Bonnet's theorem.

Lemma 7.10. *Let f be a parametrization with parameters u, v at some point $p \in S$ of regular surface S and let K be the Gaussian Curvature of S. It follows that*
$\frac{D}{\partial v}\frac{D}{\partial u} f_u - \frac{D}{\partial u}\frac{D}{\partial v} f_u = K(f_u \times f_v) \times f_u.$

Lemma 7.11. *Let h be a differentiable mapping and let V be a differentiable vector field along h. It follows that, if $K(s,t)$ is the curvature of S at the point $h(s,t)$:*
$\frac{D}{\partial t}\frac{D}{\partial s} V - \frac{D}{\partial s}\frac{D}{\partial t} V = K(s,t)(\frac{\partial h}{\partial s} \times \frac{\partial h}{\partial t}) \times V.$

Proposition 7.12. *Let H be a proper orthogonal variation of a geodesic γ parametrized by the arc length s. Let $V = \frac{\partial dh}{\partial dt}$ be the variatioal vector field of h. Then*
$L''(0) = \int_0^t (\mid \frac{D}{\partial s} V \mid^2 - K \mid V \mid^2) ds$
where K is the Gaussian curvature of S at $\gamma(s) = h(s, 0)$.

Proof. By proposition 2, we get
$L'(t) = \int_0^t \frac{\frac{D}{\partial s}\frac{\partial h}{\partial t} \cdot \frac{\partial h}{\partial s}}{(\frac{\partial h}{\partial s} \cdot \frac{\partial h}{\partial s})^{\frac{1}{2}}} ds.$
If we let $t \in (-\delta, \delta)$, then by Lemma 1 we get,
$L''(t) = \frac{(\frac{D}{\partial s}\frac{\partial h}{\partial t} \cdot \frac{\partial h}{\partial s})^2}{(\frac{\partial h}{\partial s} \cdot \frac{\partial h}{\partial s})^{\frac{3}{2}}} ds$
Because γ is a geodesic and because the variation is orthogonal, $(\frac{\partial h}{\partial s} \cdot \frac{\partial h}{\partial t} = 0$ for $t = 0$, it follows that,
$L''(0) = \int_0^t \frac{d}{dt}(\frac{D}{\partial s}\frac{\partial h}{\partial t} \cdot \frac{\partial h}{\partial s}) ds.$
when $s = 0$. Through some additional algebra, one can get $L''(0)$ into the form,
$L''(0) = \int_0^t (\mid \frac{D}{\partial s} V(s) \mid^2 - K \mid V(s) \mid^2) ds.$
□

This expression $L''(0)$ is called the second variation of arc length.

8. Bonnet's Theorem

Now, with all of the needed background, we can now prove Bonnet's Theorem.

Theorem 8.1. *Let the Gaussian curvature K of a complete surface S satisfy the condition*
$K \geq \delta > 0$
It follows that S is compact and the diameter ρ of S satisfies the inequality
$\rho \leq \frac{\pi}{\sqrt{\delta}}.$

Proof. Let S be complete. By Hopf-Rinow, it follows that, given two points $p, q \in S$, there exists a minimal geodesic γ joining p to q. We will show that the length $l = (p, q)$ of this geodesic satisfies the inequality

$l \leq \frac{\pi}{\sqrt{\delta}}$.

By contradiction, assume that $l > \frac{\pi}{\sqrt{\delta}}$. Consider a variation of the geodesic γ such that w_0 is a unit vector of $T_{\gamma(0)}(S)$ such that $w_0 \cdot \gamma'(0) = 0$ and let $w(s)$ be the parallel transport of w_0 along γ. It follows that $\mid w(s) \mid = 1$ and that $w(s) \cdot \gamma'(s) = 0$. Now consider the vector field $V(s)$ such that

$V(s) = w(s) sin(\frac{\pi}{l} s)$.

Because $V(0) = V(1) = 0$ and $V(s) \cdot \gamma'(s) = 0$, it follows that $V(s)$ determines a propoper, orthogonal variation of γ. By proposition 4,

$L''_V(0) = \int_0^t (\frac{\pi^2}{l^1} cos^2(\frac{\pi}{l} s) - K sin^2(\frac{\pi}{l} s)) ds$

Because we assumed $l > \frac{\pi}{\sqrt{\gamma}}$, it follows that $K \geq \gamma > \frac{\pi^2}{l^2}$, and we get

$L''_V(0) < \frac{\pi^2}{l^2} \int_0^t cos(\frac{2\pi}{l} s) ds = 0$.

Thus, there exists a variation of γ where $L''(0) < 0$. Because γ is a geodesic, it follows that any variation of γ should have $L''(0) \geq 0$. That is a contradiction.

Therefore, because, for any $p, q \in S$, $d(p, q) \leq \frac{\pi}{\sqrt{\delta}}$, we get that S is bounded and that its diameter is $\rho \leq \frac{\pi}{\sqrt{\gamma}}$. Because S is complete and bounded, it is compact. □

That is Bonnet's theorem; that any Surface with a positive Gaussian curvature that is larger than a surface's radius is compact.

9. Applications of Bonnet's Theorem

While not as famous as the Theorema Egregium or Gauss-Bonnet's theorem, Bonnet's theorem is an important statement that is used in Differential Geometry. Bonnet's theorem is used in multiple different global theorems of curves, it can be used to show that any complete surface with a vanishing Gaussian curvature is a plane or a cylinder, and it can be used to prove Hilbert's theorem: a theorem that states there is no complete regular surface in R^3 with constant negative Gaussian curvature. Proofs for these theorems can be found in sections 5-8 and 5-11 of do Carmo. It is clear that Bonnet's theorem both stands on its own and is also important for proving other theorems in Differential Geometry. Especially when one includes other mathematical uses for Bonnet's theorem, one sees an important theorem that deserves attention.

10. Acknowledgements

I would like to thank my mentor Andrew Geng for guiding me through this paper and answering my numerous questions. I would also like to thank Chase and the late do Carmo for their works which I used to create this paper. I would like to thank Professor May both for creating and running the REU and for guidance in writing the paper.

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