HYPERBOLIC PLANE AS A PATH METRIC SPACE

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ABSTRACT. We will study the quantities that are invariant under the action of Möb($\mathbb{H}$) on hyperbolic plane, namely, the length of paths in hyperbolic plane. From these invariant elements, we construct an invariant notion of hyperbolic distance on $\mathbb{H}$ and explore some of its basic properties. And we will prove that Möb($\mathbb{H}$)= Isom($\mathbb{H}, d_{\mathbb{H}}$).

CONTENTS

1. Introduction and Basic Definitions

Hyperbolic geometry is a non-Euclidean geometry in which the parallel postulate in Euclidean geometry does not hold. In hyperbolic geometry, for any given line L and a point p not in L, there exist at least two distinct lines through p that are parallel to L. Hyperbolic space is a a model for hyperbolic geometry. It could be two-dimensional or higher. And there are several models for hyperbolic space, for instance the hyperboloid model, the Klein model, the Poincaré ball model and the Poincaré half space model. For any two of the above models, there exist transformations between them that preserve geometric properties. In this paper, we use the upper half plane to model the hyperbolic plane and we will explore geometries in the hyperbolic plane. The *upper half-plane* model is defined to be $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. And we will refer to the real axis by $\mathbb{R}$.

We are going to define a *hyperbolic line* in $\mathbb{H}$ in terms of Euclidean lines and circles in $\mathbb{C}$.

Definition 1.1. A *hyperbolic line* in $\mathbb{H}$ is either the intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ perpendicular to $\mathbb{R}$ or the intersection of $\mathbb{H}$ with a Euclidean circle centered on $\mathbb{R}$.

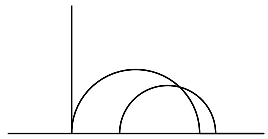

Figure 1.1: Hyperbolic lines

Proposition 1.2. *For each pair of distinct points, p and q, in $\mathbb{H}$, there exists a unique hyperbolic line l through p and q.*

Proof. There are two cases to consider. Case 1: Suppose $\mathrm{Re}(p) = \mathrm{Re}(q)$. Then the Euclidean line $L = \{z \in \mathbb{C} \mid\} \mathrm{Re}(z) = \mathrm{Re}(p)\}$ passes through p and q and is perpendicular to $\mathbb{R}$. By the uniqueness of the Euclidean line, we see that $l = \mathbb{H} \cap L$ is the unique hyperbolic line through p and q.

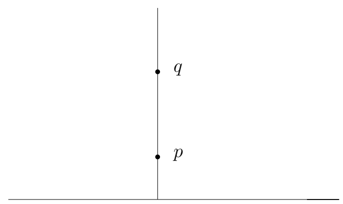

Figure 1.2: Hyperbolic line determined by point p and q

Case 2: Suppose $\mathrm{Re}(p) \neq \mathrm{Re}(q)$. Then let L_{pq} be the Euclidean line segment joining p and q and let K be the perpendicular bisector of L_{pq} . Then K intersects $\mathbb{R}$ at a unique point c and $|c-p| = |c-q|$. Then let C be a Euclidean circle centered at c with radius $|c-p|$. So $l = \mathbb{H} \cap C$ is the desired unique hyperbolic line.

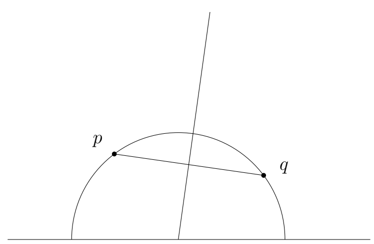

Figure 1.3: Hyperbolic line determined by point p and q

$\square$

Hyperbolic geometry behaves differently from Euclidean geometry of parallel lines, although the definitions of *parallel* for Euclidean lines and hyperbolic lines are same.

Definition 1.3. If two hyperbolic lines are disjoint, then they are *parallel*.

Theorem 1.4. *Let l be a hyperbolic line in $\mathbb{H}$ and let p be a point in $\mathbb{H}$ not in l, there exist infinitely many distinct hyperbolic lines through p that are parallel to l.*

Proof. There are two cases to consider. First, suppose that l is contained in a Euclidean line L and p is a point in $\mathbb{H}$ not in l. As p is not in L, p is on Euclidean line K that is parallel to L. Since L is perpendicular to $\mathbb{R}$, K is perpendicular to R. Then $K \cap \mathbb{H}$ is a hyperbolic line through p and parallel to l. We can construct another such hyperbolic line by taking a point x in $\mathbb{R}$ and let a be the hyperbolic line through p and x. k and a are distinct because $\mathrm{Re}(x) \neq \mathrm{Re}(p)$. And we can construct infinitely many such hyperbolic lines because there are infinitely points on $\mathbb{R}$ between Ł and K.

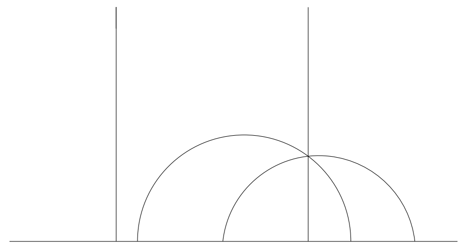

Figure 1.4: Parallel hyperbolic lines

The second case is that l is contained in a Euclidean circle M. As p is in $\mathbb{H}$ but not in L , there is a Euclidean circle N through p and concentric to M . As concentric circles are disjoint, $N \cap \mathbb{H}$ is a hyperbolic line through p and parallel to l. To construct another such hyperbolic line, take any point x on $\mathbb{R}$ between M and N. Let a be the hyperbolic line through p and x. Then a is disjoint from l. As there are infinitely many points between M and N, we can construct infinitely many hyperbolic lines through p and parallel to l.

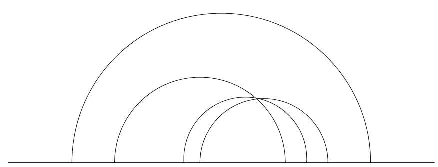

Figure 1.5: Parallel hyperbolic lines

□

To determine a reasonable group of transformations in $\mathbb{H}$ that take hyperbolic lines to hyperbolic lines, we need to unify the two types of hyperbolic lines. By stereographic projection, a Euclidean circle can be obtained from a Euclidean line by adding a single point.

Definitions 1.5. By stereographic projection, we can construct *Riemann sphere* $\bar{\mathbb{C}}$. As a set of points, the Riemann sphere is the union of the complex plane $\mathbb{C}$ with a point not in $\mathbb{C}$, which we denote by ∞, which means $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We also define the *extended real axis* $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. A circle in the Riemann sphere $\bar{\mathbb{C}}$ is either a Euclidean circle in $\mathbb{C}$ or a Euclidean line in $\mathbb{C}$ with $\{\infty\}$.

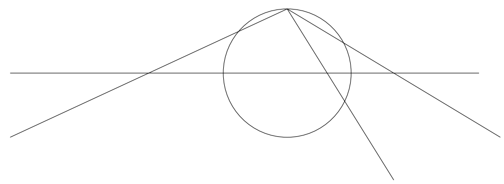

Figure 1.6: Stereografic projection

There is a class of continuous functions from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$ that are especially well behaved.

Definitions 1.6. A function $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous.

The homeomorphisms of $\bar{\mathbb{C}}$ are the transformations of $\bar{\mathbb{C}}$ that are of most interest of us. We denote the group of homeomorphisms by $\mathrm{Homeo}(\bar{\mathbb{C}}) = \{f : \bar{\mathbb{C}} \to \bar{\mathbb{C}} \mid f$ is a homeomorphism$\}$. By definition, the inverse of a homeomorphism is also a homeomorphism and the composition of two homeomorphisms is also a homeomorphism. And the identity homeomorphism is the function $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ given by $f(z) = z$. So $\mathrm{Homeo}(\bar{\mathbb{C}})$ is a group.

2. The General Möbius Group and Möb($\mathbb{H}$)

As every hyperbolic line in $\mathbb{H}$ is contained in a circle in $\bar{\mathbb{C}}$, in order to determine the transformations of $\mathbb{H}$ that take hyperbolic lines to hyperbolic lines, we first determine the group of homeomorphisms of $\bar{\mathbb{C}}$ taking circles $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$. We let $\mathrm{Homeo}^C(\bar{\mathbb{C}})$ denote the subset of $\mathrm{Homeo}(\bar{\mathbb{C}})$ that contains homeomorphisms of $\bar{\mathbb{C}}$ taking circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$.

Definition 2.1. A *Möbius transformation* is a function $m : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of the form
$$m(z) = \frac{az+b}{cz+d}$$
where $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$. We denote the set of all Möbius transformations by Möb^+.

Lemma 2.2. *The element f of Homeo($\bar{\mathbb{C}}$) defined by*
$$f(z) = az + b \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty, \text{ where } a, b \in \mathbb{C} \text{ and } a \neq 0,$$
is an element of Homeo^C($\bar{\mathbb{C}}$).

Proof. Given a circle in $\bar{\mathbb{C}}$ $A = \{z \in \bar{\mathbb{C}} : \alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0\}$, where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$ and $\alpha = 0$ if and only if A is a Euclidean line. We want to show that if $z \in A$, then $w = az + b$ satisfy a similar equation. We first consider the case $\alpha = 0$. Since $w = az + b$, we can substitute $z = \frac{1}{a}(w - b)$ into the equation obtaining
$$\begin{aligned}\beta z + \bar{\beta}\bar{z} + \gamma &= \beta\frac{1}{a}(w-b) + \bar{\beta}\overline{\frac{1}{a}(w-b)} + \gamma \\ &= \frac{\beta}{a}w + \frac{\bar{\beta}}{a}\bar{w} - 2\mathrm{Re}(\frac{\beta}{a}b) + \gamma = 0\end{aligned}$$

This shows that w satisfies an equation of an Euclidean line. So f takes Euclidean lines to Euclidean lines.

If $\alpha \neq 0$, we have

$$\begin{aligned}\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma &= \alpha \frac{1}{a}(w-b)\overline{\frac{1}{a}(w-b)} + \beta\frac{1}{a}(w-b) + \bar{\beta}\overline{\frac{1}{a}(w-b)} + \gamma \\ &= \frac{\alpha}{|a|^2}(w-b)\overline{(w-b)} + \frac{\beta}{a}(w-b) + \frac{\bar{\beta}}{a}\overline{(w-b)} + \gamma \\ &= \frac{\alpha}{|a|^2}|w + \frac{\bar{\beta}a}{\alpha} - b|^2 + \gamma - \frac{|\beta|^2}{\alpha} = 0\end{aligned}$$

So w satisfy an equation of a Euclidean circle. Hence f takes circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$.

$\square$

Lemma 2.3. *The element g of Homeo($\bar{\mathbb{C}}$) defined by*

$$g(z) = \frac{1}{z} \text{ for } z \in \mathbb{C} - \{0\},\ g(0) = \infty, \text{ and } g(\infty) = 0,$$

is an element of Homeo^C($\bar{\mathbb{C}}$).

The proof of Lemma 2.3 is similar to Lemma 2.2. In fact, Möb$^+$ is generated by f and g.

Theorem 2.4. *Let m be a Möbius transformation $m(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. If $c = 0$, then $m(z) = \frac{a}{d}z + \frac{b}{d}$. If $c \neq 0$, then $m(z) = f \circ g \circ h(z)$, where $h(z) = c^2(z) + cd$ for $z \in \mathbb{C}$ and $h(\infty) = \infty$, $g(z) = \frac{1}{z}$ for $z \in \mathbb{C} - \{0\}$, $g(0) = \infty$, and $g(\infty) = 0$, and $f(z) = -(ad - bc)z + \frac{a}{c}$ for $z \in \mathbb{C}$ and $f(\infty) = \infty$.*

The proof of Theorem 2.4 is just direct calculation. By Theorem 2.4 and Lemmas 2.2 and 2.3, we have that every Möbius transformation is an element of Homeo$^{\text{C}}$($\bar{\mathbb{C}}$), as it is the composition of functions in Homeo$^{\text{C}}$($\bar{\mathbb{C}}$). So we have the following theorem.

Theorem 2.5. *Möb$^+$ $\subset$ Homeo^C($\bar{\mathbb{C}}$).*

To extend Möb$^+$ to a larger group, we consider the *complex conjugation*, $C : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $C(z) = \bar{z}$ and $C(\infty) = \infty$, which is not an element of Möb $^+$ but is an element of Homeo$^{\text{C}}$($\bar{\mathbb{C}}$).

Lemma 2.6. *The function $C : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $C(z) = \bar{z}$ and $C(\infty) = \infty$ is an element of Homeo^C($\bar{\mathbb{C}}$).*

The proof of Lemma 2.6 is similar to the proof of lemma 2.2.

Definition 2.7. The *general Moöbius group* is the group generated by Möb$^+$ and the function $C : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $C(z) = \bar{z}$ and $C(\infty) = \infty$. Let Möb denote the general Möbius group.

Theorem 2.8. *Möb = Homeo^C($\bar{\mathbb{C}}$)*

Sketch of Proof:
By Lemma 2.8, we have that every element of Möb is an element of Homeo$^{\text{C}}$($\bar{\mathbb{C}}$). Then we need to show that Homeo$^{\text{C}}$($\bar{\mathbb{C}}$) $\subset$ Möb. Let f be an element of Homeo$^{\text{C}}$($\bar{\mathbb{C}}$)

and let n be the Möbius transformation taking the triple $(f(0), f(1), f(\infty))$ to the triple $(0, 1, \infty)$. So $n \circ f$ takes $(0, 1, \infty)$ to $(0, 1, \infty)$. As $\bar{\mathbb{R}}$ is the circle in $\bar{\mathbb{C}}$ determined by $(0, 1, \infty)$ and $n \circ f$ takes circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$, $n \circ f(\bar{\mathbb{R}}) = \bar{\mathbb{R}}$. Therefore $n \circ f(\mathbb{H})$ is either the upper-half plane or the lower-half plane. In the former case, set $m = n$. In the latter case, set $m = C \circ n$. So $m \circ f$ is an element of Möb such that $m \circ f(0) = 0$, $m \circ f(1) = 1$, $m \circ f(\infty) = \infty$, and $m \circ f(\mathbb{H}) = \mathbb{H}$. We show that $m \circ f$ is the identity by constructing a dense set of points in $\bar{\mathbb{C}}$, each of which is fixed by $m \circ f$. Hence $f = m^{-1}$ is an element of Möb. This completes the sketch of the proof of the Theorem 2.9.

Notations 2.9.
Möb($\bar{\mathbb{R}}$) $= \{m \in \text{Möb} \mid m(\bar{\mathbb{R}}) = \bar{\mathbb{R}}\}$
Möb($\mathbb{H}$) $= \{m \in \text{Möb} \mid m(\mathbb{H}) = \mathbb{H}\}$

Theorem 2.10. *Every element of Möb($\bar{\mathbb{R}}$) is of one of the following form:*
1. $m(z) = \frac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*;*
2. $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*;*
3. $m(z) = \frac{az+b}{cz+d}$ *with* a, b, c, d *purely imaginary and* $ad - bc = 1$*;*
4. $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ *with* a, b, c, d *purely imaginary and* $ad - bc = 1$*;*

Theorem 2.11. *Every element of Möb($\mathbb{H}$) is of one of the following form:*
1. $m(z) = \frac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*;*
2. $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ *with* a, b, c, d *purely imaginary and* $ad - bc = 1$*;*

One useful property of Möb($\mathbb{H}$) is the transitivity property.

Theorem 2.12. *Möb ($\mathbb{H}$) acts transitively on $\mathbb{H}$, which means for every pair of points x and y in $\mathbb{H}$, there exists an element m of Möb($\mathbb{H}$) such that $m(x) = y$.*

Proof. Notice that if for each point x in $\mathbb{H}$, there exists an element m of Möb($\mathbb{H}$) such that $m(x) = y_0$ for some point in $\mathbb{H}$, then Möb($\mathbb{H}$) acts transitively on $\mathbb{H}$. Because for any two points x and z in $\mathbb{H}$, if $m_1(x) = y_0 = m_2(z)$, then $m_2^{-1} \circ m_1(x) = z$. So we just need to show that for any point x in $\mathbb{H}$, there exists an element m of Möb($\mathbb{H}$) such that $m(x) = i$. Write $x = a + bi$, where $a, b \in \mathbb{R}$ and $b > 0$. We first move x to the imaginary axis using $\gamma_1(z) = z - a$, so $m_1(x) = bi$. Then we apply $\gamma_2(z) = \frac{1}{b}z$ to $\gamma_1(x)$, so $\gamma_2(\gamma_1(x)) = i$. By Theorem 2.11, γ_1 and γ_2 are elements of Möb($\mathbb{H}$), hence so is $\gamma_2 \circ \gamma_1$. □

Theorem 2.13. *Möb($\mathbb{H}$) acts transitively on the set L of hyperbolic lines in $\mathbb{H}$.*

Proof. It is suffice to show that for each hyperbolic line l, there exists an element m of Möb($\mathbb{H}$) that takes l to the imaginary axis I. Suppose z is a point in l. By Theorem 2.11, there is an element m of Möb($\mathbb{H}$) such that $m(z) = i$. Let ϕ be the angle between the two hyperbolic lines I and $m(l)$, measured from I to $m(l)$. For each θ, the Möbius transformation $n_\theta(z) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$ lies in Möb($\mathbb{H}$) and fixes i. And the angle between I and $n_\theta(I)$ at i, measured from I to $n_\theta(I)$ is 2θ. So if we take $\theta = \frac{\phi}{2}$, then $m(l) = n_\theta(I)$, so $n_\theta^{-1} \circ m(l) = I$.

□

3. Length and Distance in $\mathbb{H}$

In this section, we will derive a means of measuring lengths of paths in $\mathbb{H}$ that is invariant under the action of Möb, expressed as an invariant element of arc-length.

Theorem 3.1. *For every positive constant c, the element of arc-length*

$$\frac{c}{Im(z)}|dz|$$

on $\mathbb{H}$ is invariant under the action of Möb($\mathbb{H}$)

Proof. Let $\rho = \frac{c}{\mathrm{Im}(z)}$. We need to show that

$$\int_a^b \rho(f(t))|f'(t)|dt = \int_a^b \rho(\gamma \circ f(t))|\gamma'(f(t))||f'(t)|dt$$

for every piecewise C^1 path $f : [a, b] \to \mathbb{H}$ and every element γ of Möb($\mathbb{H}$), which also can be written as $\int_a^b (\rho(f(t)) - \rho(\gamma \circ f(t))|\gamma'(f(t))|)|f'(t)|dt = 0$. So for an element γ of Möb($\mathbb{H}$), let $\mu_\gamma(z) = \rho(z) - \rho(\gamma(z))|\gamma'(z)$. We just need to show that

$$\int_f \mu_\gamma(z)|dz| = \int_a^b \mu_\gamma(f(t))|f'(t)|dt = 0.$$

Now consider how μ_γ behaves under composition with elements of Möb($\mathbb{H}$). Let γ and ϕ be elements of Möb($\mathbb{H}$). Then

$$\begin{aligned}
\mu_{\gamma\circ\phi} &= \mu(z) - \mu((\gamma \circ \phi(z))|(\gamma \circ \phi)'(z)| \\
&= \mu(z) - \mu((\gamma \circ \phi(z))|(\gamma'(\phi)'(z))||\phi'(z)| \\
&= \mu(z) - \mu(\phi(z))|\phi'(z)| + \mu(\phi(z))|\phi'(z)| - \mu((\gamma \circ \phi(z))|(\gamma'(\phi)'(z))||\phi'(z)| \\
&= \mu_\phi(z) + \mu_\gamma(\phi(z))|\phi'(z)|.
\end{aligned}$$

So if $\mu_\gamma = 0$ for every γ in the generating set of Möb($\mathbb{H}$), then μ_γ=0 for every element γ of Möb($\mathbb{H}$). A generating set for Möb($\mathbb{H}$) is

$$\{\gamma_1 = z + b, \gamma_2 = az, \gamma_3 = -\frac{1}{z}, \gamma_4 = -\bar{z}|a, b \in \mathbb{R}, a > 0\}.$$

Then

$$\mu_{\gamma_1} = \rho(z) - \rho(\gamma_1(z))|\gamma_1'(z)| = \rho(z) - \rho(z + b) = 0.$$

Similarly for γ_2, γ_3 and γ_4, one can check that $\mu_{\gamma_i} = 0$. This completes the proof. □

We can not determine the specific value of c using only Möb($\mathbb{H}$). For easy calculation, we set c =1.

Definition 3.2. For a piecewise C^1 path $f : [a, b] \to \mathbb{H}$, we define the *hyperbolic length* of f to be $\mathrm{length}_{\mathbb{H}}(f)$=$\int_a^b \frac{1}{\mathrm{Im}(f(t))}|f'(t)|\mathrm{d}t$.

Definition 3.3. Suppose X is a metric space with metric d. We say that (X, d) is a *path metric space* if for each pair of x and y in X,

$$d(x, y) = \inf\{\mathrm{length}(f)|f \text{ is a path connecting } x \text{ and } y\},$$

and there exists a distance-realizing path f such that $d(x, y) = length(f)$.

Definition 3.4. For each pair of points x and y of $\mathbb{H}$, let $\Gamma[x,y]$ denote the set of all piecewise C^1 paths $f : [a,b] \to \mathbb{H}$ with $f(a) = x$ and $f(b) = y$. We define the function $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, $d_{\mathbb{H}}(x,y) = \inf\{\text{length}_{\mathbb{H}}(f) | f \in \Gamma[x,y]\}$. We refer to $d_{\mathbb{H}}(x,y)$ as the *hyperbolic distance* between x and y.

Proposition 3.5. *For every element γ of Möb($\mathbb{H}$) and for every pair x and y of points of $\mathbb{H}$, we have that $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(\gamma(x),\gamma(y))$.*

Proof. Let $f : [a,b] \to \mathbb{H}$ be a path in $\Gamma[x,y]$ and γ be in Möb($\mathbb{H}$). As $\text{length}_{\mathbb{H}}(f)$ is invariant under Möb($\mathbb{H}$), we have $\text{length}_{\mathbb{H}}(f)$=$\text{length}_{\mathbb{H}}(\gamma \circ f)$. And observe that for every path $f \in \Gamma[x,y]$, $\gamma \circ f \in \Gamma[x,y]$. So

$$\begin{aligned} d_{\mathbb{H}}(\gamma(x),\gamma(y)) = \inf\{\ &\text{length}_{\mathbb{H}}(g) | g \in \Gamma[\gamma(x),\gamma(y)]\} \\ \leq\ &\inf\{\text{length}_{\mathbb{H}}(\gamma \circ f) | f \in \Gamma[x,y]\} \\ \leq\ &\inf\{\text{length}_{\mathbb{H}}(f) | f \in \Gamma[x,y]\} \\ &= d_{\mathbb{H}}(x,y). \end{aligned}$$

Now consider γ^{-1}, which is also an element of Möb($\mathbb{H}$). Then

$$\begin{aligned} d_{\mathbb{H}}(x,y) &= \inf\{\text{length}_{\mathbb{H}}(f) | f \in \Gamma[x,y]\} \\ \leq\ &\inf\{\text{length}_{\mathbb{H}}(\gamma^{-1} \circ g | g \in \Gamma[\gamma(x),\gamma(y)]\} \\ \leq\ &\inf\{\text{length}_{\mathbb{H}}(g) | g \in \Gamma[\gamma(x),\gamma(y)]\} \\ &= d_{\mathbb{H}}(\gamma(x),\gamma(y)). \end{aligned}$$

Therefore, $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(\gamma(r),\gamma(y))$. □

Theorem 3.6. $(\mathbb{H}, d_{\mathbb{H}})$ *is a path metric space.*

Proof. First, we show that $(\mathbb{H}, d_{\mathbb{H}})$ is a metric.

As $\text{length}_{\mathbb{H}}(f) = \int_a^b \frac{1}{\text{Im}(f(t)} |f'(t)| \text{d}t$ is nonnegative for all path $f \in \Gamma(x,y)$, their infimum $d_{\mathbb{H}}(x,y)$ is nonnegative.

Let $f : [a,b] \to \mathbb{H}$ be a path in $\Gamma[x,y]$, and let $h : [a,b] \to [a,b]$ given by $h(t) = a + b - t$. Then $f \circ h \in \Gamma[y,x]$ and

$$\text{length}_{\mathbb{H}}(f \circ h) = \int_a^b \frac{1}{\text{Im}(f(h(t))} |f'(h(t)||h'(t)| \text{d}t = \int_a^b \frac{1}{\text{Im} f(s) |f'(s)|} \text{d}s = \text{length}_{\mathbb{H}}(f).$$

So every path in $\Gamma[x,y]$ can be reparametrized to become a path in $\Gamma[y,x]$. So $\{\text{length}_H(f) | f \in \Gamma[x,y]\} = \{\text{length}_H(g) | g \in \Gamma[y,x]\}$, hence they have the same infimum. So $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(y,x)$.

Now we prove the triangle equality by contradiction. Suppose that x, y, z are points in $\mathbb{H}$ such that $d_{\mathbb{H}}(x,z) > d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)$. Then let

$$\epsilon = d_{\mathbb{H}}(x,z) - d_{\mathbb{H}}(x,y) - d_{\mathbb{H}}(y,z) > 0.$$

As $d_{\mathbb{H}}(x,y) = \inf\{\text{length}_H(f) | f \in \Gamma[x,y]\}$, there exists a path f in $\Gamma[x,y]$ with $\text{length}_{\mathbb{H}}(f) - d_{\mathbb{H}}(x,y) < \frac{1}{2}\epsilon$. Similarly there exists a path g in $\Gamma[x,y]$ with $\text{length}_{\mathbb{H}}(g) - d_{\mathbb{H}}(y,z) < \frac{1}{2}\epsilon$. Let $h \in \Gamma[x,z]$ be the concatenation of f and g. Then

$$d_{\mathbb{H}}(x,z) < \text{length}_{\mathbb{H}}(h) = \text{length}_{\mathbb{H}}(f) + \text{length}_{\mathbb{H}}(g) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) + \epsilon,$$

which contradicts the construction of ϵ. So the triangle equality is satisfied by $d_{\mathbb{H}}$.

Then we show that there exists a path in $\mathbb{H}$ realizing the hyperbolic distance between any pair of points of $\mathbb{H}$, which implies that $d_{\mathbb{H}}(x,y) > 0$ if $x \neq y$. So let x, y be a pair of distinct point in $\mathbb{H}$ and let l be the hyperbolic line passing through x and y. By Theorem 2.13, there exists an element $\gamma \in$Möb($\mathbb{H}$) so that $\gamma(l)$ is the positive imaginary axis in $\mathbb{H}$. We can write $\gamma(x) = \mu i$ and $\gamma(y) = \lambda i$ with $\mu < \lambda$. By Proposition 3.5, $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(\gamma(x), \gamma(y))$. So we just need to prove that there exist a distance-realizing path from μi to λi.We begin by calculating the hyperbolic length of a specific path, $f_0 : [\mu, \lambda] \to \mathbb{H}$ defined by $f_0(t) = it$. So length$_{\mathbb{H}}(f_0) = \int_{\mu}^{\lambda} \frac{1}{t} dt = \ln \frac{\lambda}{\mu}$.Then we show that length$_{\mathbb{H}}(f_0) \leq$length$_{\mathbb{H}}(f)$ for any path $f \in \Gamma[\mu i, \lambda i]$. Let $f : [a,b] \to \mathbb{H}$ be defined by $f(t) = x(t) + y(t)i$. Consider the path $g : [a,b] \to \mathbb{H}$ defined by $g = y(t)i$, so $g \in \Gamma[\mu i, \lambda i]$. Since $(x'(t))^2 \geq 0$, length$_{\mathbb{H}}(g) \leq$length$_{\mathbb{H}}(f)$. So we reduced ourselves to showing that if $g \in \Gamma[\mu i, \lambda i][$ is of the form $g(t) = y(t)i$, then length$_{\mathbb{H}}(f_0) \leq$length$_{\mathbb{H}}(g)$. The image of g is the hyperbolic line segment joining αi and βi with $\alpha \leq \mu \leq \lambda \leq \beta$. Define $h : [\alpha, \beta] \to \mathbb{H}$ by $h(t) = it$. Then we have that length$_{\mathbb{H}}(f_0) = \ln \frac{\lambda}{\mu} \leq \ln \frac{\beta}{\alpha} =$ length$_{\mathbb{H}}(h)$. Then we can write $g = h \circ (h^{-1} \circ g)$, where $h^{-1} \circ g : [a,b] \to [\alpha, \beta]$ is a surjective function. Therefore length$_{\mathbb{H}}(h) \leq$length$_{\mathbb{H}}(g)$. This completes the argument that length$_{\mathbb{H}}(f_0) \leq$length$_{\mathbb{H}}(f)$ for every path f in $\Gamma[\mu i, \lambda i]$. Therefore, for every pair of distinct points x and y in $\mathbb{H}$, let l be the hyperbolic line through x and y, and let γ be the element of Möb($\mathbb{H}$) taking l to the imaginary axis with $\gamma(x) = \mu i$ and $\gamma(y) = \lambda i$ where $\mu < \lambda$. Define $f_0 : [\mu, \lambda] \to \mathbb{H}$ by $f_0(t) = it$. Then $\gamma^{-1} \circ f_0$ is a distance-realizing path in $\Gamma[x, y]$.

□

4. Isometries in $\mathbb{H}$

Definition 4.1. An *isometry* of a metric space (X, d) is a homeomorphism f of X that preserves distance.

Proposition 4.2. *Let x, y and z be distinct points in $\mathbb{H}$. Then $d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z)$ if and only if y is contained in the hyperbolic line segment joining x to z.*

Proof. By Theorem 2.11 and Theorem 2.12, there exists an element m of Möb($\mathbb{H}$) such that $m(x) = i$ and $m(z) = \beta i$, $1 < \beta$. Then $d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(i, \beta i) = \ln \beta$. Write $m(y) = a + bi$. If y is contained in the hyperbolic line segment joining x and z, then $m(y)$ lies on the hyperbolic line segment joining $m(x)$ and $m(y)$. So $a = 0, 1 < b < \beta$ and $d_{\mathbb{H}}(x,y) = \ln b$ and $d_{\mathbb{H}}(y,z) = \ln \frac{\beta}{b}$. Hence $d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z)$. Suppose now that y is not contained in the hyperbolic line segment joining x to z. There are two cases.

First case: $a = 0$, in other words, $m(y)$ lies on the imaginary axis. Then we have either $0 < b < 1$ or $b > \beta$. If $0 < b < 1$, then $d_{\mathbb{H}}(x,y) = -\ln b]$ and $d_{\mathbb{H}}(y,z) = \ln \beta - \ln b.]$ As $\ln b < 0$,

$$d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z) - 2\ln b > d_{\mathbb{H}}(x,z).$$

If $b > \beta$, then $d_{\mathbb{H}}(x,y) = \ln b$ and $d_{\mathbb{H}}(y,z) = \ln b - \ln \beta$. As $\ln b > d_{\mathbb{H}}(x,z)$,

$$d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) = 2\ln b - d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(x,z).$$

The second case is that $a \neq 0$. Then $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(i, a+bi) < d_{\mathbb{H}}(i, bi)$ and $d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(a+bi, \beta i) > d_{\mathbb{H}}(bi, \beta i)$. If $1 < b < \beta$, then

$$d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(i,\beta) = d_{\mathbb{H}}(i,bi) + d_{\mathbb{H}}(bi,\beta i) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z).$$

Similarly if $0 < b < 1$, then

$$d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(x,z) - 2\ln b = d_{\mathbb{H}}(i,bi) + d_{\mathbb{H}}(bi,\beta i) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z).$$

If $b > \beta$, then

$$d_{\mathbb{H}}(x,z) < 2\ln b - d_{\mathbb{H}}(x,z) = d_{\mathbb{H}}(i,bi) + d_{\mathbb{H}}(bi,\beta i) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z).$$

□

Proposition 4.3. *Let f be a hyperbolic isometry and let l be a hyperbolic line, then $f(l)$ is also a hyperbolic line.*

Proof. By Proposition 4.2, if y lies in the hyperbolic line segment l_{xz}, then $d_{\mathbb{H}}(x,y)+ d_{\mathbb{H}}(y,z) = d_{\mathbb{H}}(x,z)$. Since f preserves distance, $d_{\mathbb{H}}(f(x), f(y)) + d_{\mathbb{H}}(f(y), f(z)) = d_{\mathbb{H}}(f(x), f(z))$. So $f(y)$ lies in the hyperbolic segment $l_{f(x)f(y)}$ joining $f(x)$ to $f(y)$, hence $f(l_{xy})= l_{f(x)f(y)}$. As a hyperbolic line can be expressed as a nested union of hyperbolic line segments, we have that hyperbolic isometries take hyperbolic lines to hyperbolic lines. □

Proposition 4.4. *Let (z_1, z_2) and (w_1, w_2) be pairs of distinct points of $\mathbb{H}$. If $d_{\mathbb{H}}(z_1, z_2) = d_{\mathbb{H}}(w_1, w_2)$, then there exists an element m of Möb($\mathbb{H}$) satisfying $m(z_1) = w_1$ and $m(z_2) = w_2$.*

Proof. We can construct p and q of Möb($\mathbb{H}$) such that $q(z_1) = p(w_1) = i$ and $q(z_2) = p(w_2) = e^{d_{\mathbb{H}}(z_1,z_2)}i$. So $m = p^{-1} \circ q$ satisfies $m(z_1) = w_1$ and $m(z_2) = w_2$. □

Theorem 4.5. *Let Isom($\mathbb{H}, d_{\mathbb{H}}$) denote the group of isometries of ($\mathbb{H}, d_{\mathbb{H}}$), Then Isom($\mathbb{H}, d_{\mathbb{H}}$)=Möb($\mathbb{H}$).*

Proof. By Proposition 3.6, we have that Möb($\mathbb{H}$)$\subset$ Isom($\mathbb{H}, d_{\mathbb{H}}$). Then we show the opposite inclusion using Proposition 4.3. Let f be a hyperbolic isometry and for each pair of points p and q of $\mathbb{H}$, let l_{pq} be the hyperbolic line segment joining p to q. Let l be the perpendicular bisector of l_{pq}, in other words, $l = \{x \in \mathbb{H} | d_{\mathbb{H}}(x,p) = d_{\mathbb{H}}(x,q)\}$. Then $f(l)$ is the perpendicular bisector of $f(l_{pq}) = l_{f(p)f(q)}$. Now let x and y be points on the positive imaginary axis I in $\mathbb{H}$ and let H be one of the half-planes in $\mathbb{H}$ determined by I. By Proposition 4.4, there exists an element m of Möb($\mathbb{H}$) such that $m(f(x)) = x$ and $m(f(y)) = y$. Since $m \circ f$ fixes x and y, $m \circ f$ takes I to I. If necessary, replace m by $B \circ m$ with $B(z) = -\bar{z}$ to have m also take $\mathbb{H}$ to $\mathbb{H}$. Let z be a point on I. As z is uniquely determined by two hyperbolic distance $d_{\mathbb{H}}(x,z)$ and $d_{\mathbb{H}}(y,z)$ and as $m \circ f$ preserves hyperbolic distances, we have that $m \circ f$ fixes every point of I. Let w be any point in $\mathbb{H}$ but not in I and let l be the hyperbolic line through w perpendicular to I. So l is the hyperbolic line contained in the Euclidean circle with Euclidean center 0 and Euclidean radius $|w|$. Let z be the intersection of l and I. As l is the perpendicular bisector of some hyperbolic line segment in I and as $m \circ f$ fixes every point in I, $m \circ f(l)$=l. As $m \circ f$ fixes z, as $d_{\mathbb{H}}(z,w) = d_{\mathbb{H}}(m \circ f(z), m \circ f(w)) = d_{\mathbb{H}}(z, m \circ f(w))$, and as $m \circ f$ preserves the two half-planes determined by I, we have that $m \circ f$ fixes w.

Therefore $m \circ f$ is the identity. So $f = m^{-1}$, hence f is an element of Möb($\mathbb{H}$). Therefore Isom($\mathbb{H}, d_{\mathbb{H}}$)=Möb($\mathbb{H}$).

□

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